

EQUIVARIANT AND BOTT-TYPE SEIBERG-WITTEN FLOER HOMOLOGY: PART II

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ABSTRACT. We construct equivariant and Bott-type Seiberg-Witten Floer homology and cohomology for 3-manifolds, in particular rational homology spheres, and prove their diffeomorphism invariance.

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1. INTRODUCTION

This paper is the sequel of [24], and Part II of our series of papers on Bott-type and equivariant Seiberg-Witten Floer homology. As explained in the introduction to Part I (see [24]), our goal is to construct diffeomorphism invariants for 3-manifolds, in particular rational homology spheres, which are based on the Seiberg-Witten theory and the Floer homology theory. The major problem which we have to resolve in the process of constructions is that of reducible Seiberg-Witten points and reducible transition trajectories. In [24], we constructed the Bott-type Seiberg-Witten Floer homology and cohomology and provided part of the proof for their invariance, which is based on the key technique “spinor perturbation”. (The remaining part will be presented in [26].) Besides, a number of basic ingredients of analytical or geometrical nature for our theory were presented.

In the present paper, we construct equivariant Seiberg-Witten Floer homologies and cohomologies and prove their invariance. A fundamental ingredient in our constructions (both in Part I and the present Part II) is a theory on the Morse-Floer-Bott flow complex along with its projection to critical submanifolds. Part of this theory was presented in Part I. In this paper, we complete the theory. Most of these results were obtained when [23], the earlier version of [24], was written in 1996. To arrange more balanced size of our papers, we place them in the present Part II rather than in Part I.

We continue with the set-up, notations and terminologies in [24]. In particular, we work in the framework of the Seiberg-Witten theory on a given rational homology sphere Y . The following is a brief account of the above topics.

Our basic equations are the (perturbed) 3-dimensional Seiberg-Witten equation $\mathbf{sw}_{\lambda,H}(a, \phi) = 0$ on Y , i.e.

$$(1.1) \quad \begin{cases} *_Y F_a + \langle e_i \cdot \phi, \phi \rangle e^i &= \nabla H(a), \\ \not\partial_a \phi + \lambda \phi &= 0 \end{cases}$$

with perturbations H (holonomy) and λ (a real number), and the (perturbed) Seiberg-Witten trajectory equation (flow equation) $\mathbf{SW}_{\lambda,H}(A, \Phi) = 0$ on $X = Y \times \mathbb{R}$, i.e.

$$(1.2) \quad \begin{cases} \frac{\partial a}{\partial t} - *_Y F_a - d_Y f - \langle e_i \cdot \phi, \phi \rangle e^i &= \nabla H(a), \\ \frac{\partial \phi}{\partial t} + \not\partial_a \phi + \lambda \phi + f \phi &= 0 \end{cases}$$

with $A = a + f dt$, $\Phi = \phi$, cf. [24]. Our basic geometrical objects, on which we build our theories, are the moduli space \mathcal{R}^0 of based gauge classes of Seiberg-Witten points (solutions of (1.1)) and the moduli spaces of Seiberg-Witten trajectories or flow lines (solutions of (1.2)). The fundamental functional involved is the Seiberg-Witten type Chern-Simons functional \mathbf{cs} on the configuration space $\mathcal{A}(Y) \times \Gamma(Y)$ and its based gauge quotient $\mathcal{B}^0 = \mathcal{A}(Y) \times \Gamma(Y) / \mathcal{G}^0$.

Equivariant Constructions: Singular Version

The basic idea is to couple the configuration space $\mathcal{A}(Y) \times \Gamma(Y)$ with a space \mathbf{S} which has a free S^1 action. If we first divide out by the based gauge group \mathcal{G}^0 , then what we do is to couple the based gauge quotient \mathcal{B}^0 with \mathbf{S} . We have the diagonal action of S^1 on the product $\mathcal{B}^0 \times \mathbf{S}$. The Chern-Simons functional is extended to the product $\mathcal{B}^0 \times \mathbf{S}$ in the trivial way, namely $\mathbf{cs}(\alpha, \mathbf{s}) = \mathbf{cs}(\alpha)$. It is invariant under the S^1 action, and hence descends to the quotient $\mathcal{B}^0 \times_{S^1} \mathbf{S}$. Its critical submanifolds are given by the S^1 quotient $\mathcal{R}^0 \times_{S^1} \mathbf{S}$ of the *stablized moduli space* $\mathcal{R}^0 \times \mathbf{S}$. We use *generalized cubical singular chains and cochains* on them to build our basic chains and cochains. The boundary operator will then be constructed by utilizing the classical boundary operator along with *stablized Morse-Floer-Bott flow complex*, i.e. the *stablized moduli spaces* of (based gauge classes of) Seiberg-Witten trajectories, or rather the S^1 quotient of this complex, which we call the *quotient stablized Morse-Floer-Bott flow complex*. Here, the *stablization* again means multiplying by \mathbf{S} . For each choice of \mathbf{S} , we obtain in this fashion an equivariant Seiberg-Witten Floer homology and cohomology.

The most obvious choices for \mathbf{S} are the odd dimensional spheres S^{2n+1} , $n = 0, 1, \dots$, and the infinite dimensional sphere S^∞ . Indeed, these choices are natural from the viewpoint of the classical equivariant homology and cohomology construction due to Borel [7], and the viewpoint of equivariant Morse theory. (But we were not led to these choices this way. Instead, we arrived at them in an intuitive way.) Recall that for a topological space \mathcal{X} with an action by a group G , its G -equivariant homology and cohomology as introduced by Borel [7] are defined to be the homology and cohomology of the homotopy quotient space $EG \times_G \mathcal{X}$, where EG denotes a contractible space with a free G action (thus $BG = EG/G$ is the classifying space of G), and G acts on $EG \times \mathcal{X}$ by the diagonal action. Now, if \mathcal{B}^0 were a compact manifold and the Chern-Simons functional were an ordinary equivariant Morse function, then our construction with $\mathbf{S} = S^\infty = EG^1$ would reproduce the

classical equivariant homology and cohomology of \mathcal{B}^0 , namely the homology and cohomology of the homotopy quotient $S^\infty \times_{S^1} \mathcal{B}^0$. Of course, the goal of our constructions is precisely to produce new invariants which are intimately tied to the smooth structure of the manifold Y , and the basic point is to bring the special features of the Seiberg-Witten and Chern-Simons geometry into play. Indeed, our new invariants are profoundly different from the classical equivariant theory. The main analytic reason for it is the fact that the spectrum of the Hessian of the Chern-Simons functional is infinite in both positive and negative directions.

In the classical situation with e.g. $G = S^1$, the theory built on $ES^1 = S^\infty$ can be approximated by the theories based on S^{2n+1} . In spirit, our equivariant Seiberg-Witten Floer theories based on S^{2n+1} are also approximations of our theory based on S^∞ . We shall refer to these equivariant theories (with S^{2n+1} or S^∞) as the *singular version* of equivariant Seiberg-Witten Floer homology and cohomology.

Instead of chains and cochains on the quotient space $\mathcal{R}^0 \times \mathbf{S}$, we can also use invariant cochains on the stablized space $\mathcal{R}^0 \times \mathbf{S}$. This is the formulation presented in [23], an earlier version of [24], which we call the *stable equivariant theory*. It is essentially equivalent to the above (quotient) equivariant construction. (At that time we already formulated the quotient version, but didn't include it in the paper. The reason was that we could only handle $\mathbf{S} = S^1$, and hence the advantage of the quotient version could not be exploited, see the relevant discussion below.) The precise relations between the equivariant and stable equivariant theories, and between the S^{2n+1} theory and the S^∞ theory, will be computed in Part III [26]. We acknowledge a helpful conversation with K. Fukaya on these concepts.

Invariance

The most fundamental issue about these theories is their diffeomorphism invariance. We establish their invariance by utilizing the *stablized transition Morse-Floer-Bott flow complex* and its S^1 quotient, the *quotient stablized transition Morse-Floer-Bott flow complex*. (In the proof, certain extensions of the transition flow allowing additional parameters are needed. We mean to include them here.) Note that here stablization is not simply taking product with \mathbf{S} . Instead, \mathbf{S} is built into the stable transition flow equation in a nontrivial fashion. As in [24], one has to kill reducible transition trajectories in order to achieve transversality. For this purpose, we employ a key *gauge equivariant spinor perturbation* for the stable transition flow equation. It is a modification of the spinor perturbation for the transition flow equation introduced in [24], which is equivariant only with respect to based gauges. Note that for the Bott-type theory in [24], the spinor perturbation of the transition flow equation provides only part of the invariance proof. However, the gauge equivariant spinor perturbation of the stable transition flow equation suffices fully for establishing the invariance of the equivariant theories.

We would like to mention that in [23], besides the spinor perturbation for the transition flow equation which is based gauge equivariant, we already constructed a gauge equivariant spinor perturbation for the stable transition flow equation. However, it worked only for the case $\mathbf{S} = S^1$. Now we have a way to extend our construction to the cases $\mathbf{S} = S^{2n+1}$, $n = 1, 2, \dots$ and $\mathbf{S} = S^\infty$, and hence we are able to complete our equivariant theory. The completion of the singular version also allows us to establish the invariance of the de Rham version and the Cartan version, which will be addressed below.

There is another subtle point in the invariance proof we would like to mention. In the second step of the proof, we utilize a certain extension IIE_ϵ of the transition flow equation (ϵ is a parameter) to establish the isomorphism equation $F_{-*}^+ \cdot F_{+*}^- = Id$. It turns out that IIE_0 coincides with the original Seiberg-Witten flow equation. The delicate feature here is that all the moduli spaces $\mathcal{M}_T^0(S_\alpha \times_{S^1} \mathbf{S}, S_\beta \times_{S^1} \mathbf{S}) \equiv \mathcal{M}_T^0(S_\alpha, S_\beta) \times_{S^1} \mathbf{S}$ with $\alpha = \beta$ as well as $\alpha \neq \beta$ are involved, but no time translation is allowed. At a first glance, it appears that we are running into trouble with compactification, for we had to use the time translation quotient $\underline{\mathcal{M}}_T^0(S_\alpha \times_{S^1} \mathbf{S}, S_\beta \times_{S^1} \mathbf{S})$ in order to obtain the compactification $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} \mathbf{S}, S_\beta \times_{S^1} \mathbf{S})$. This trouble is resolved when we realize that here a different compactification scheme is at play. As a consequence, the contribution of the moduli spaces $\mathcal{M}_T^0(S_\alpha \times_{S^1} S^1, S_\beta \times_{S^1} S^1)$ with $\alpha \neq \beta$ to the induced chain map is trivial, and the remaining moduli spaces $\mathcal{M}_T^0(S_\alpha \times_{S^1} S^1, S_\alpha \times_{S^1} S^1)$ give rise to the identity on the right hand side of the equation $F_{-*}^+ \cdot F_{+*}^- = Id$. Note that this feature is irrelevant in Floer's situation [14], where only moduli spaces corresponding to $\alpha = \beta$ are involved.

de Rham Version

Instead of generalized cubical singular chains and cochains on the quotient space $\mathcal{R}^0 \times_{S^1} \mathbf{S}$, we can also use differential forms on it. The resulting theory will be called the *de Rham version of equivariant Seiberg-Witten Floer homology and cohomology*. The construction of the de Rham version uses in an essential way the fibration property of the projections of the quotient stablized Morse-Floer-Bott flow complex. To establish the invariance of this version, one might want to use the quotient stablized transition Morse-Floer-Bott flow complex. However, one has to perform perturbations to kill reducible transition trajectories in order to bring this flow complex into general position. The key perturbations we use for proving the invariance of the singular version achieve these goals, but the resulting projections of the transition flow complex may not be fibrations. In general, we think that it is unlikely to find suitable perturbations without destroying the fibration property of the projections. Thus, it seems that any attempt of directly proving the invariance of the de Rham version in terms of the transition flow is doomed to fail.

We shall establish the invariance of the de Rham version by showing that it is isomorphic to the singular version with real coefficients. The tool for establishing the isomorphism is the spectral sequence induced from the index filtration associated with the Chern-Simons functional.

We shall also construct the de Rham version of the Bott-type Seiberg-Witten Floer homology and cohomology and show that it is isomorphic to the singular version constructed in Part I [24] with real coefficients. The relations between the equivariant theories and the Bott-type theories will be computed in Part III.

Cartan Version

Besides Borel's homotopy quotient construction, there is another well-known classical construction of equivariant cohomology based on equivariant Lie algebra valued differential forms. It is due to H. Cartan [8] and isomorphic to the Borel construction with real coefficients. In [4], equivariant instanton Floer homology and cohomology were constructed by utilizing Cartan's model. Similar constructions can be carried out in the Seiberg-Witten set up. A number of aspects of this

construction have independently been presented in [19]. We present an account of this construction, which is based on our theory of the Morse-Floer-Bott flow complex along with its projection to critical submanifolds. This version will be referred to as the *Cartan version of equivariant Seiberg-Witten Floer homology and cohomology*. A major difference between this version and the singular version as well as the de Rham version of equivariant theory is that here the chains and cochains live on the moduli space \mathcal{R}^0 rather than the stablized moduli space $\mathcal{R}^0 \times S$ or its S^1 quotient.

Like the de Rham version, the construction of the Cartan version uses in an essential way the fibration property of the projection of the *unstablized* Morse-Floer-Bott flow complex. Similar to the situation of the de Rham version, one runs into troubles when trying to prove its invariance directly by using the transition Morse-Floer-Bott flow complex. Indeed, the trouble here goes deeper than the fibration property. In order to deal with equivariant differential forms, one needs to retain the full gauge equivariance of the transition flow equation when perturbing it into transversal position. Unfortunately, this is impossible to achieve. (Although we can construct full gauge equivariant transversal perturbations for the *stable* transition flow equation as discussed above.) Otherwise, one would be able to use the constructions to show that the conventional Serberg-Witten Floer homology is invariant, which, according to [12], [9] and [18], fails to be true for homology spheres in general. (The paper [19] misses this crucial point. Indeed, in [19] the issue of transversal perturbation for the transition flow is not addressed.)

We prove the invariance of the Cartan version by showing that it is isomorphic to the de Rham version. The isomorphism will be established by utilizing the spectral sequence induced from the index filtration associated with the Chern-Simons functional.

Exponential Convergence

We present a detailed treatment of the issues of exponential decay and convergence of Seiberg-Witten trajectories. The convergence analysis contains four themes. First, uniform pointwise smooth estimates hold for a sequence of temporal Seiberg-Witten trajectories of uniformly bounded energy, provided that they are in appropriate gauges. Using this we obtain local convergence of the trajectories. But energy might be lost near infinities during convergence, which is not captured by the limit of local convergence. We manage to get additional limits and show that they together capture all energies. This is the second theme and quite similar in spirit to the convergence arguments about pseudo-holomorphic curves presented in [27]. The third theme is to show that the convergence of the relevant piece of the trajectory to each limit actually holds in exponential norm. This is an indispensable point. The fourth theme is to show that the endpoints of the limit temporal trajectories match each other. In other words, the endpoint at the positive time infinity of one limit trajectory is identical to the endpoint of the next limit trajectory at the negative time infinity. Indeed, this property lies at the heart of the analysis of the structures of the compactified moduli spaces of trajectories.

We would like to point out that in the above analysis it is important to deal with temporal trajectories, i.e. use temporal gauges. The first reason is that the key analytical properties of the trajectories can be best utilized in temporal gauges. (We also need to use local Columb gauges and “slice gauges”, namely gauges determined along each Y -slice, as auxiliary tools.) The second reason is that the projections of

our moduli spaces of trajectories have to be defined in terms of temporal gauges, see [24]. (Indeed they are called *temporal projections*.) For this reason, it is crucial to establish the endpoint matching property for limit temporal trajectories. Further relevant discussions are given below.

Structure Near Infinity

The convergence analysis discussed above is one pillar in the analysis of the structures of the (compactified) moduli spaces of trajectories. The goal of this analysis is to show that these moduli spaces are compact smooth manifolds with corners. The other pillar is a procedure of gluing trajectories, i.e. deforming piecewise trajectories into smooth trajectories. Gluing arguments have widely been used in gauge theories, in particular in Floer's instanton homology theory. There are delicate new aspects in our situation, however. First, as mentioned above and explained in [24], the endpoint projections have to be defined in terms of the temporal model. The appropriate objects as compactification limits of trajectories are then the *consistent piecewise trajectories* and *consistent multiple temporal trajectory classes*, where “consistent” means that the temporal projections of the pieces in a piecewise trajectory (or multiple trajectory class) match each other. Our convergence argument shows that trajectories in suitable gauges indeed converge to consistent piecewise trajectories. Now we need to show the converse, namely consistent piecewise trajectories can be deformed back into smooth trajectories. This is done by a carefully designed gluing process. The crucial new feature here is the consistent condition. Indeed, conventional gluing set-ups in the literature do not take care of the temporal projections which are the key in the constructions of the equivariant and Bott-type theories. We carry out the gluing process in the set-up of the temporal model $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$, which we prefer because of its canonical formulation and global features. We also sketch this process in the set-up of the fixed-end model $\underline{\mathbf{M}}^0(p, q)$.

The second delicate aspect is this. Merely gluing piecewise trajectories into smooth trajectories falls far short from establishing the structure of smooth manifolds with corners. What is needed is suitable coordinate charts based on the gluing construction. To show that the gluing construction indeed yields the desired charts, we need to show the local diffeomorphism (in the interior) and homeomorphism property of the gluing construction. For this purpose, it is crucial to derive careful estimates for various sizes involved in the gluing construction. (There are treatments of this issue in the literature with various degrees of details, but our situation is very different.)

2. SINGULAR VERSIONS OF EQUIVARIANT THEORY

Recall the set-up in [24]: we consider a 3-dimensional rational homology sphere Y , along with a Riemannian metric h and a $spin^c$ structure c on Y . (For simplicity, we only present the case that Y is connected. Our arguments work equally well for Y with more than one components.) Moreover, we consider a pair of Y -generic parameters (π, λ) , namely generic parameters for the perturbed Seiberg-Witten equation (1.1). Recall also that for $\alpha \in \mathcal{R}$, S_α means the lift of α to \mathcal{R}^0 , and that $S_i = \cup\{S_\alpha : \alpha \in \mathcal{R}, \mu(\alpha) = i\}$. (The index μ was defined in [24, Section 5].) We have $\mathcal{R}^0 = \cup_i S_i$.

Finally, we recall that (see [24, Section 7]) to a topological space \mathcal{X} and coefficient group \mathbf{G} , we associate the complex $(C_*(\mathcal{X}; \mathbf{G}), \partial_O)$ of generalized cubical singular chains, and the complex $(C^*(\mathcal{X}; \mathbf{G}), \partial^*)$ of generalized cubical singular cochains.

Equivariant Seiberg-Witten Floer Homology and Cohomology

Let \mathbf{S} be a topological space with a free S^1 action. We choose \mathbf{S} to be either the odd dimensional euclidean spheres $S^{2n-1} \subset \mathbb{C}^n$ or S^∞ . The action of S^1 on them is multiplication by unit complex numbers. Note that there are two well-known models for S^∞ . One is the unit sphere in a separable Hilbert space. The other is the unit sphere in the space \mathbb{C}^∞ whose elements are sequences (c_1, c_2, \dots) of complex numbers with finitely many nonzero entries. The second model is precisely the direct limit of S^{2n-1} as n approaches infinity. We can use either one to carry out our constructions. We choose the second one, because it is more convenient.

We have the diagonal action of S^1 on $\mathcal{R}^0 \times \mathbf{S}$ and the corresponding quotient $\mathcal{R}^0 \times_{S^1} \mathbf{S} = \cup_i S_i \times_{S^1} \mathbf{S}$. (Recall that the action of S^1 on \mathcal{R}^0 is via the identification of S^1 with constant gauges.) Fix a coefficient group \mathbf{G} . We introduce our equivariant chain complex $C_*^{equ} = C_*^{equ; \mathbf{G}}$ and cochain complex $C_{equ}^* = C_{equ; \mathbf{G}}^*$,

$$C_k^{equ} = \oplus_{i+j=k} C_j(S_i \times_{S^1} \mathbf{S}; \mathbf{G}), C_{equ}^k = \oplus_{i+j=k} C^j(S_i \times_{S^1} \mathbf{S}; \mathbf{G}).$$

We have $C_*^{equ} = C_*(\mathcal{R}^0 \times_{S^1} \mathbf{S}; \mathbf{G})$, $C_{equ; \mathbf{G}}^* = C^*(\mathcal{R}^0 \times_{S^1} \mathbf{S}; \mathbf{G})$.

To construct the desired boundary operators for the equivariant chain and cochain complexes, we consider the moduli spaces of consistent multiple temporal Seiberg-Witten trajectory classes $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ introduced in [24, Section 6] and their stabilization $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta) \times \mathbf{S}$. We write the latter as $\underline{\mathbf{M}}_T^0(S_\alpha \times \mathbf{S}, S_\beta \times \mathbf{S})$. Indeed, a pair (u, \mathbf{s}) with u a Seiberg-Witten trajectory can be considered as a solution of the trivially stabilized Seiberg-Witten trajectory equation, namely the Seiberg-Witten trajectory equation with a non-appearing, trivial variable \mathbf{s} . With this set-up, the moduli spaces $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ used in [24] for the Bott-type theory are replaced by $\underline{\mathbf{M}}_T^0(S_\alpha \times \mathbf{S}, S_\beta \times \mathbf{S})$. For this reason, we adopt the notation $[u, \mathbf{s}]_0^T$ for $([u]_0^T, \mathbf{s})$. (Recall that the subscript 0 refers to based gauges, while the superscript T means temporal.)

We have the diagonal action of S^1 on these moduli spaces. (Recall that the action of S^1 on the Seiberg-Witten trajectories is via the identification of S^1 with constant gauges.) Their quotients $\underline{\mathbf{M}}_T^0(S_\alpha \times \mathbf{S}, S_\beta \times \mathbf{S})/S^1$, which we shall denote by $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} \mathbf{S}, S_\beta \times_{S^1} \mathbf{S})$, now play the role of Morse-Floer-Bott flow complex for the equivariant chain complex and cochain complexes. We call them the *quotient stabilized Morse-Floer-Bott flow complex*. Since the temporal projections π_+ and π_- in [24, Section 6] are equivariant with respect to the S^1 action, we can define the temporal projections for the quotient stabilized Morse-Floer-Bott flow complex as follows

$$\pi_+([u, \mathbf{s}]_0^T) = ([\pi_+(u)]_0, \mathbf{s}), \pi_-([u, \mathbf{s}]_0^T) = ([\pi_-(u)]_0, \mathbf{s}).$$

With these preparations, it is easy to carry over the construction of the boundary operators $\partial_{Bott}, \partial_{Bott}^*$ in [24, Section 7] to the present situation to produce a boundary operator $\partial_{equ} : C_k^{equ} \rightarrow C_{k-1}^{equ}$ and the corresponding coboundary operator $\partial_{equ}^* : C_{equ}^k \rightarrow C_{equ}^{k+1}$. Note that in the case $\mathbf{S} = S^\infty$, the construction of these operators involves for each generalized cubical singular chain a fixed finite dimensional sphere in S^∞ , and hence is no different from the construction in the case $\mathbf{S} = S^{2n+1}$. This is because that every generalized singular cube has compact image in $\mathcal{R}^0 \times_{S^1} S^\infty$. (In general, for similar reasons, all of our constructions in the case $\mathbf{S} = S^\infty$ involve the same analysis as those in the case S^{2n+1} .) Similar to $(\partial_{Bott})^2 = 0, (\partial_{Bott}^*)^2 = 0$, we have $(\partial_{equ})^2 = 0, (\partial_{equ}^*)^2 = 0$. Hence we can introduce the following equivariant Seiberg-Witten Floer homology and cohomology

Definition 2.1. We define (for each given coefficient group \mathbf{G} and $spin^c$ structure c)

$$FH_{equ*}^{SW} = H_*(C_*^{equ}, \partial_{equ}), FH_{equ*}^{SW*} = H^*(C_*^{equ}, \partial_{equ}^*).$$

If we need to indicate the coefficient group \mathbf{G} and the $spin^c$ structure c , we can write e.g. $FH_{equ; \mathbf{G}*}^{SW}(c)$.

As in [24], we consider the index filtration

$$\mathcal{F}_*^{equ} = \dots \mathcal{F}_k^{equ} \subset \mathcal{F}_{k+1}^{equ} \dots$$

and its dual filtration \mathcal{F}_{equ}^* for our equivariant chain and cochain complexes, where

$$\mathcal{F}_k^{equ} = \oplus_{j \leq k} C_*(S_j \times_{S^1} \mathbf{S}; \mathbf{G}),$$

$$\mathcal{F}_{equ}^k = \oplus_{j \geq k} C^*(S_j \times_{S^1} \mathbf{S}; \mathbf{G}).$$

We have

Theorem 2.2. *The equivariant index filtration \mathcal{F}_*^{equ} induces a spectral sequence $E(equ)_{**}^*$ converging to FH_{equ*}^{SW} such that*

$$(2.1) \quad E(equ)_{ij}^1 \cong H_j(S_i \times_{S^1} \mathbf{S}; \mathbf{G}).$$

*The dual filtration induces a spectral sequence $E(equ, dual)_{**}^*$ converging to FH_{equ}^{SW*} such that*

$$(2.2) \quad E(equ, dual)_{ij}^1 \cong H^j(S_i \times_{S^1} \mathbf{S}; \mathbf{G}).$$

The proof is similar to the proof for [24, Theorem 7.8], hence we omit it.

Stable Equivariant Seiberg-Witten Floer Homology and Cohomology

We consider the same \mathbf{S} and fix a coefficient group \mathbf{G} as before. The action of S^1 on $\mathcal{R}^0 \times \mathbf{S}$ induces an action on generalized cubical singular chains in $\mathcal{R}^0 \times \mathbf{S}$: for generators $(\Delta, f) \otimes \mathbf{g}$ and $g \in S^1$, we have $g^*((\Delta, f) \otimes \mathbf{g}) = (\Delta, g^*f(\cdot)) \otimes \mathbf{g}$. Passing to quotients, we obtain an action of S^1 on $C_*(\mathcal{R}^0 \times \mathbf{S})$. A j -cochain class $\omega \in C^*(\mathcal{R}^0 \times \mathbf{S}; \mathbf{G})$ is called *invariant*, provided that $\omega(g^*\sigma) = \omega(\sigma)$ for all $\sigma \in C_*(\mathcal{R}^0 \times \mathbf{S}; \mathbb{Z})$ and $g \in S^1$.

We define $C_{inv}^j(\mathcal{R}^0 \times \mathbf{S})$ to be the group of invariant j -cochain classes on $\mathcal{R}^0 \times \mathbf{S}$ and introduce the stable equivariant cochain complex C_{sequ}^* ,

$$C_{sequ}^k = \oplus_{i+j=k} C_{inv}^j(S_i \times \mathbf{S}).$$

Using the stablized moduli spaces $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta) \times \mathbf{S}$ we can easily carry over the construction of the coboundary operator ∂_{Bott}^* to obtain a boundary operator $\partial_{sequ}^* : C_{sequ}^k \rightarrow C_{sequ}^{k+1}$ for the stable equivariant complex. The gauge equivariance of the projections π_\pm ensures that ∂_{sequ} indeed produces invariant cochains out of invariant cochains. We have $\partial_{sequ}^2 = 0$.

Definition 2.3. The stable equivariant Seiberg-Witten Floer cohomology FH_{sequ}^{SW*} (for the given coefficient group \mathbf{G} and $spin^c$ structure c) is defined to be the cohomology of the cochain complex $(C_{sequ}^*, \partial_{sequ}^*)$. The stable equivariant Seiberg-Witten Floer homology FH_{sequ*}^{SW} is defined to be its dual homology.

We have a stable equivariant index filtration

$$\mathcal{F}^*(sequ) = \cdots \mathcal{F}^k(sequ) \subset \mathcal{F}^{k-1}(sequ) \subset \cdots,$$

$$\mathcal{F}^k(sequ) = \oplus_{j \geq k} C_{inv}^*(S_k \times \mathbf{S}).$$

There is a corresponding dual filtration. The corresponding spectral sequences are similar to those in Theorem 2.3.

Remark 2.4. We can also formulate a stable version of the Bott-type theory in [24], whose information is contained in the Bott-type theory. We leave the details to the reader.

3. INVARIANCE

The purpose of this section is to prove the following theorem.

Theorem 3.1. *The singular version of equivariant Seiberg-Witten Floer homology and cohomology are diffeomorphism invariants up to shifting isomorphisms.*

We start with

Definition 3.2. Consider $\mathbf{S} = S^{2n-1}$ or S^∞ . Let the group of based gauges $\mathcal{G}_{3,loc}^0 \equiv \mathcal{G}_{3,loc}^0(X)$ (recall $X = Y \times \mathbb{R}$) act on $(\mathcal{A}_{2,loc}(X) \times \Gamma_{2,loc}^+(X)) \times \mathbf{S}$ in the following fashion:

$$g^*(u, \mathbf{s}) = (g^*u, \mathbf{s}),$$

where $g \in \mathcal{G}_{3,loc}^0$, $u \in \mathcal{A}_{2,loc} \times \Gamma_{2,loc}^+$ and $\mathbf{s} \in \mathbf{S}$. On the other hand, we have the diagonal action of S^1 on $(\mathcal{A}_{2,loc} \times \Gamma_{2,loc}^+) \times \mathbf{S}$:

$$g^*(u, \mathbf{s}) = (g^*u, g^{-1}\mathbf{s}),$$

where $g \in S^1$. Combining these two actions we then obtain an action of the full group of gauges $\mathcal{G}_{3,loc}$ on $(\mathcal{A}_{2,loc} \times \Gamma_{2,loc}^+) \times \mathbf{S}$.

We first consider the case $\mathbf{S} = S^1$, which allows a special treatment, simpler than the general one.

Definition 3.3. We define a smooth vector field Z_e on $(\mathcal{A}_{2,loc} \times \Gamma_{2,loc}^+) \times S^1$ as follows

$$Z_e(u, s) = sZ(u),$$

where Z is the vector field given by [24, Definition 8.3].

The following lemma is readily proved

Lemma 3.4. Z_e is equivariant with respect to the action of $\mathcal{G}_{3,loc}$.

For two metrics h_{\pm} on Y and two pairs of Y -generic parameters $(\pi_{\pm}, \lambda_{\pm})$ for h_{\pm} respectively, we have their interpolations $h(t), \pi(t), \lambda(t)$ as given in [24, Section 8]. We introduce the following stable version of the (perturbed) transition trajectory equation ([24, (8.1)]) for $A = a + f dt, \Phi = \phi$ and $s \in S^1$. Its solutions will be called *stable transition trajectories*.

$$(3.1) \quad \begin{cases} \frac{\partial a}{\partial t} &= *_Y F_a + d_Y f + \langle e_i \cdot \phi, \phi \rangle e^i + \nabla H_{\pi(t)}(a) + b_0, \\ \frac{\partial \phi}{\partial t} &= -\partial_a \phi - \lambda(t) \phi + Z_e((A, \Phi), s), \end{cases}$$

where $s \in S^1$, and b_0 is a perturbation form. Several operations in this equation (such as the Hodge star) depend in an obvious way on the interpolating metric $h(t)$ at time t . Note that this equation can be written in the following fashion:

$$(3.2) \quad \begin{cases} F_A^+ = & \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j \\ & + (\nabla H_{\pi(t)}(a) + b_0) \wedge dt + *((\nabla H_{\pi(t)}(a) + b_0) \wedge dt), \\ D_A \Phi = & -\lambda \frac{\partial}{\partial t} \cdot (\Phi - Z_e((A, \Phi), s), \end{cases}$$

where X is endowed with the warped-product metric determined by $h(t)$ and the standard metric on \mathbb{R} .

As a consequence of Lemma 3.4 and the construction of Z_e we have

Lemma 3.5. *The equation (3.1) is invariant under the action of $\mathcal{G}_{3,loc}$. Moreover, it has no reducible solution.*

We have moduli spaces of temporal stable transition trajectories $\mathcal{M}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$ with $\alpha_- \in \mathcal{R}_-$ and $\alpha_+ \in \mathcal{R}_+$, where \mathcal{R}_- and \mathcal{R}_+ are the moduli spaces of gauge equivalence classes of Seiberg-Witten points for the parameters (h_-, π_-, λ_-) and (h_+, ϕ_+, λ_+) respectively. They are analogous to the moduli spaces $\mathcal{M}_T^0(S_{\alpha_-}, S_{\alpha_+})$ in [24, Section 8]. We also have the moduli spaces of stable, consistent multiple temporal transition trajectory classes (SCMTC), $\mathbf{M}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$. A SCMTC is an element $([u_1, s_1]_0^T, \dots, [u_k, s_k]_0^T)$ in $\mathcal{M}_T^0(S_{\alpha_0^-} \times S^1, S_{\alpha_1^-} \times S^1) \times_{S_{\alpha_1^-}} \dots \mathcal{M}_T^0(S_{\alpha_{m-1}^-} \times S^1, S_{\alpha_m^+} \times S^1) \times_{S_{\alpha_m^+}} \dots \mathcal{M}_T^0(S_{\alpha_{k-1}^+} \times S^1, S_{\alpha_k^+} \times S^1)$. The time translation acts on all portions except the distinguished one $[u_m, s_m]_0^T$. The quotient of $\mathbf{M}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$ under the resulting \mathbb{R}^k -action is denoted by $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)$. All these are similar to the constructions and notations in [24, Section 8 and Section 6].

We have the following analogue of [24, Theorem 8.9].

Theorem 3.6. *The moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$ are compact Hausdorff spaces with respect to the topology induced from smooth piecewise exponential convergence (cf. [24, Definition 6.14]). Moreover, for generic b_0 , the following hold for all $\alpha_- \in \mathcal{R}_-, \alpha_+ \in \mathcal{R}_+$.*

(1) $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$ has the structure of d -dimensional smooth oriented manifolds with corners, where $d = \mu_-(\alpha_-) - \mu_+(\alpha_+) + m_0 + \dim G_{\alpha_+} - \max\{\dim G_{\alpha_-}, \dim G_{\alpha_+}\} + 2$, and

with \mathcal{F}_{O_-, O_+} denoting the index of the linearized transition Seiberg-Witten operator between the two reducible Seiberg-Witten points O_-, O_+ . (The stable parameter in S^1 does not enter into the index counting. But it contributes an additional one to the dimension formula.)

(2) This structure is compatible with the natural stratification of $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$.

(3) The (canonically defined) temporal projections $\pi_{\pm} : \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1) \rightarrow S_{\alpha_{\pm}} \times S^1$ are S^1 -equivariant smooth maps. (But they may not be fibrations in general.)

Consequently, there holds

$$(3.3) \quad \begin{aligned} \partial \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1) &= (\cup_{\mu(\alpha_-) > \mu(\alpha'_-) \geq \mu(\alpha_+) - m_0} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, S_{\alpha'_-} \times S^1) \\ &\times_{(S_{\alpha'_-} \times S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha'_-} \times S^1, S_{\alpha_+} \times S^1)) \cup (\cup_{\mu(\alpha_-) \geq \mu(\alpha'_+) - m_0 > \mu(\alpha_+) - m_0} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^1, \\ &S_{\alpha'_+} \times S^1) \times_{(S_{\alpha'_+} \times S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha'_+} \times S^1, S_{\alpha_+} \times S^1)). \end{aligned}$$

Passing to S^1 -quotient, we obtain the following result.

Theorem 3.7. *The moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)$ are compact Hausdorff spaces. Moreover, for generic b_0 , the following hold for all $\alpha_- \in \mathcal{R}_-, \alpha_+ \in \mathcal{R}_+$.*

(1) $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)$ has the structure of d -dimensional smooth oriented manifolds with corners, where $d = \mu_-(\alpha_-) - \mu_+(\alpha_+) + m_0 + \dim G_{\alpha_+} - \max\{\dim G_{\alpha_-}, \dim G_{\alpha_+}\} + 1$.

(2) This structure is compatible with the natural stratification of $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)$.

(3) The (canonically defined) temporal projections $\pi_{\pm} : \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1) \rightarrow S_{\alpha_{\pm}} \times_{S^1} S^1$ are smooth maps. (But they may not be fibrations in general.)

Consequently, there holds

$$(3.4) \quad \begin{aligned} \partial \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1) &= (\cup_{\mu(\alpha_-) > \mu(\alpha'_-) \geq \mu(\alpha_+) - m_0} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, \\ &S_{\alpha'_-} \times_{S^1} S^1) \times_{(S_{\alpha'_-} \times_{S^1} S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha'_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)) \cup \\ &(\cup_{\mu(\alpha_-) \geq \mu(\alpha'_+) - m_0 > \mu(\alpha_+) - m_0} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_+} \times_{S^1} S^1) \times_{(S_{\alpha'_+} \times_{S^1} S^1)} \\ &\underline{\mathbf{M}}_T^0(S_{\alpha'_+} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)). \end{aligned}$$

Proof of Theorem 3.1 for $\mathbf{S} = S^1$. Consider the above set-up of two sets of parameters $(h_{\pm}, \pi_{\pm}, \lambda_{\pm})$. Our goal is to show that the singular version of equivariant Seiberg-Witten Floer homology (cohomology) constructed in terms of (h_-, π_-, λ_-) is isomorphic to that constructed in terms of (h_+, π_+, λ_+) . We present the case of homology, while the case of cohomology can be handled by a similar argument.

Consider the equivariant complexes C_*^{equ-} and C_*^{equ+} associated with the parameters (h_-, π_-, λ_-) and (h_+, π_+, λ_+) respectively. Let \mathbf{F}_+^- denote the collection of all projections $\pi_- : \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1) \rightarrow S_{\alpha_-} \times_{S^1} S^1$ and $\pi_+ : \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1) \rightarrow S_{\alpha_+} \times_{S^1} S^1$ (for all $\alpha_- \in \mathcal{R}_-, \alpha_+ \in \mathcal{R}_+$). We have the

subcomplex $C_*^{equ-, \mathbf{F}_+^-}$ of C_*^{equ-} consisting of \mathbf{F}_+^- -transversal chains on $\mathcal{R}_- \times_{S^1} S^1$, which are similar to $C_*^{Bott, \mathbf{F}}$ in [24, Section 7]. By the arguments in the proof of [24, Lemma 7.8], the homology $H_*(C_*^{equ-, \mathbf{F}_+^-}, \partial_{equ})$ is canonically isomorphic to $H_*(C_*^{equ-}, \partial_{equ})$.

Using the moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1)$, we construct a chain map $F_+^- : C_*^{equ-} \rightarrow C_*^{equ+}$ in the same way as the construction of the chain map F in [24, Section 8]. We denote the interpolations and perturbation parameters involved in (3.1) by $P_+^- = (h_+, \pi_+, \lambda_+, b_{0+}^-)$. Reversing the roles of (h_-, π_-, λ_-) and (h_+, π_+, λ_+) , we obtain a chain map $F_-^+ : C_*^{equ+, \mathbf{F}_-^+} \rightarrow C_*^{equ-}$. The corresponding interpolations and perturbation parameters are denoted by $P_-^+ = (h_-^+, \pi_-^+, \lambda_-^+, b_-^+)$. The associated collection of projections is denoted by \mathbf{F}_-^+ . Furthermore, let $\mathbf{F}_+^- \mathbf{F}_-^+$ denote the collection of projections π_- from $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_+} \times_{S^1} S^1) \times_{(S_{\alpha_+} \times_{S^1} S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha_+} \times_{S^1} S^1, S_{\alpha_-} \times_{S^1} S^1)$ to $S_{\alpha_-} \times_{S^1} S^1$.

Note that F_+^- has degree m_0 , and F_-^+ has degree $-m_0$. We need to show that they induce isomorphisms between the homologies. The arguments consist of three steps.

Step 1

Consider $\tau_{-R^{-1}}(P_+^-)$ and $\tau_{R^{-1}}(P_-^+)$, the time translated P_+^- and P_-^+ for $R \in (0, 1)$. Note that they coincide over $Y \times [1 - R^{-1}, -1 + R^{-1}]$. We define $P_- = P_-(R)$ to be equal to $\tau_{-R^{-1}}(P_+^-)$ over $Y \times (-\infty, -1 + R^{-1}]$ and equal to $\tau_{R^{-1}}(P_-^+)$ over $Y \times [1 - R^{-1}, \infty)$. Using P_- in (3.1) we obtain the *interpolated stablized transition equation*. For a fixed R , we denote it by IE_R . If we allow R to vary in an interval $(0, R_0]$, we denote it by $IE_{(0, R_0]}$. Roughly speaking, when $R \rightarrow 0$, IE_R breaks into the stablized transition equation with data P_+^- and the same equation with data P_-^+ .

Fix some $R_0 \in (0, 1)$. Using IE_{R_0} we obtain moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IE_{R_0}}$ along with the collection $\mathbf{F}_{IE_{R_0}}$ of projections π_- from them and their boundaries. In order to achieve transversality for these moduli spaces, we add an additional transversal perturbation form \bar{b}_0 to $P_-(R_0)$. Then a structure theorem similar to Theorem 3.7 holds for them. Using these moduli spaces we construct a chain map $F_- : C_*^{equ-, \mathbf{F}_-} \rightarrow C_*^{equ-}$ of degree zero.

On the other hand, using $IE_{(0, R_0]}$ we obtain moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IE}$. We denote the associated collection of projections by \mathbf{F}_{IE} . To achieve transversality for these moduli spaces, we add a transversal perturbation family of forms $b_0(R)$ to $P_-(R)$, such that $b_0(R_0) = \bar{b}_0$ and $b_0(R) \rightarrow 0$ as $R \rightarrow 0$. Then a structure theorem similar to Theorem 3.7 holds for them. Employing these moduli spaces we obtain a chain map $\Theta : C_*^{equ-, \mathbf{F}_{IE}} \rightarrow C_*^{equ-}$ of degree one.

We have

$$(3.5) \quad \partial \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IE} = (\cup_{\alpha''_-} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha''_-} \times_{S^1} S^1) \times_{(S_{\alpha''_-} \times_{S^1} S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha''_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)) \cup$$

$$\underline{\mathbf{M}}_T^0(S_{\alpha''_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IE} \cup$$

$$(\cup_{\alpha''_-} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha''_-} \times_{S^1} S^1)^{IE} \times_{(S_{\alpha''_-} \times_{S^1} S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha''_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)) \cup$$

$$\begin{aligned} & \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IE_{R_0}} \cup \\ & (\cup_{\alpha''_-} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha''_-} \times_{S^1} S^1) \times_{(S_{\alpha''_-} \times_{S^1} S^1)} \underline{\mathbf{M}}_T^0(S_{\alpha''_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)). \end{aligned}$$

Using this formula along with suitable orientations we can argue as in the proofs of [24, Lemma 7.5] and [24, Theorem 8.10] to infer that Θ is a chain homotopy between $F_+^+ \cdot F_+^-$ and F_- , i.e. $F_+ \cdot F_- - F_+^- = \partial_{equ} \cdot \Theta + \Theta \cdot \partial_{equ}$. Here, we restrict the chain maps F_+^-, F_+^+, F_- and Θ to the subcomplex $C_*^{equ, \mathbf{F}}$, where $\mathbf{F} = \mathbf{F}_+^- \cup \mathbf{F}_- \cup \mathbf{F}_{IE} \cup \mathbf{F}_+^+ \cup \mathbf{F}_+^-$. It follows that for the induced maps on homologies:

$$F_{+*}^- \cdot F_{-*}^+ = F_{-*}.$$

Step 2

In the equation IE_{R_0} , we have a term corresponding to b_0 in (3.1), and a term Z_e . We multiply these two terms by a real parameter ϵ and obtain a new equation IIE_ϵ . Clearly, $IIE_1 = IE_{R_0}$. On the other hand, IIE_0 is the original perturbed Seiberg-Witten trajectory equation (1.2) with some specific holonomy and λ perturbations.

We use $IIE_{[0,1]}$ to denote the equation IIE_ϵ in which ϵ is allowed to vary in $[0, 1]$. Using it we obtain moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE}$ along with the associated collection of projections \mathbf{F}_{IIE} . On the hand, we use IIE_0 to obtain moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$.

The latter moduli spaces are not the same as $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)$ used for the construction of our homology theory, although they are both built in terms of the solutions of the same equation (1.2). *This distinction is essential for our purpose.* The reason for the distinction is as follows. The moduli space $\mathcal{M}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$ is of course the same as $\mathcal{M}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)$ for the same perturbation parameters. However, since IIE_ϵ is not time translation invariant for $\epsilon > 0$, the \mathbb{R} -action on $IIE_{[0,1]}$ -trajectories is defined to be trivial. This dictates that we also define the \mathbb{R} -action on IIE_0 -trajectories to be trivial. Hence the \mathbb{R} action on $\mathcal{M}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$ is defined to be trivial. The compactification of this moduli space leads to $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$. We identify $\mathcal{M}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$ with $\underline{\mathcal{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1) \times (-1, 1)$ via the time translation and an identification of \mathbb{R} with $(-1, 1)$. Then we have for $\alpha_- \neq \alpha'_-$

$$(3.6) \quad \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0} = \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1) \times [-1, 1].$$

Note that the projections π_+ on them are constant along the fiber $[-1, 1]$.

On the other hand, we have for $\alpha'_- = \alpha_-$

$$(3.7) \quad \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_-} \times_{S^1} S^1)^{IIE_0} = \mathcal{M}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_-} \times_{S^1} S^1).$$

Indeed, here only time-independent trajectories are involved.

The transversality for $\mathcal{M}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha_-} \times_{S^1} S^1)$ follows from [24, Lemma C.1]. We incorporate an additional holonomy perturbation to achieve transversality for $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)$ for $\alpha_- \neq \alpha'_-$, cf. [24, Section 5]. Then we can incorporate additional holonomy and form perturbations to achieve transversality

for $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE}$. We have structure theorems for $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$ and $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE}$ which are similar to Theorem 3.7. We have the associated collections of projections \mathbf{F}_{IIE_0} and \mathbf{F}_{IIE} . We set $\mathbf{F} = \mathbf{F}_{IE_{R_0}} \cup \mathbf{F}_{IIE_0} \cup \mathbf{F}_{IIE}$.

Using the moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0}$ we construct a chain map $F_0 : C_*^{equ-, \mathbf{F}} \rightarrow C_*^{equ-}$. It is the sum of two maps, with one corresponding to $\alpha_- = \alpha'_-$, and one corresponding to $\alpha_- \neq \alpha'_-$. By (3.6) and the fact that the projections π_+ are constant along the fiber $[-1, 1]$, we deduce that the image of the second map is given in terms of degenerate generalized singular cubes. Hence this map is the zero map. On the other hand, it follows from (3.7) that the first map is the inclusion map (the identity map). Hence F_0 is the inclusion map.

Using the moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE}$ we construct a chain map $\tilde{\Theta} : C_*^{equ-, \mathbf{F}} \rightarrow C_*^{equ-}$. There holds

$$(3.8) \quad \begin{aligned} \partial \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE} = \\ \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE_0} \cup \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{-} \cup \\ (\cup_{\alpha''_-} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha''_-} \times_{S^1} S^1) \times_{S_{\alpha''_-} \times_{S^1} S^1} \underline{\mathbf{M}}_T^0(S_{\alpha''_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)^{IIE}) \cup \\ (\cup_{\alpha''_-} \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times_{S^1} S^1, S_{\alpha''_-} \times_{S^1} S^1)^{IIE} \times_{S_{\alpha''_-} \times_{S^1} S^1} \underline{\mathbf{M}}_T^0(S_{\alpha''_-} \times_{S^1} S^1, S_{\alpha'_-} \times_{S^1} S^1)). \end{aligned}$$

Using this equation and suitable orientations we then deduce that $\tilde{\Theta}$ is a chain homotopy between F_0 and F_- , where F_- is restricted to $C_*^{equ-, \mathbf{F}}$. It follows that $F_{-*} = Id$. Hence $F_{-*}^+ \cdot F_{+*}^- = Id$.

Step 3

A similar construction with the roles of F_- and F_+ reversed yields $F_{+*}^- \cdot F_{-*}^+ = Id$. \square

We have proved Theorem 3.1 for $\mathbf{S} = S^1$. Now we handle the general case. For higher dimensional spheres, it's no longer possible to find an equivariant S^1 valued function, hence the spinor construction given in Definition 3.2 cannot be applied. We introduce a new device. Choose Hermitian orthonormal smooth spinor fields $\Phi_k, k = 1, 2, \dots \in \Gamma^-(Y \times [-1, 1]) \equiv \Gamma(W^-|_{Y \times [-1, 1]})$ with supports contained in $Y \times (-1, 1)$. (See [24] for the definition of the spinor bundles W^+ and W^- .) Let e_k be the vector in \mathbb{C}^∞ whose k -th entry is 1, and all other entries are zero. The assignment $e_k \rightarrow \Phi_k$ determines an Hermitian complex linear embedding Ψ from \mathbb{C}^∞ into $\Gamma^-(Y \times [-1, 1])$. In particular, it is equivariant under the S^1 action, which is multiplication by unit complex numbers.

Definition 3.8. Consider the global slice \mathcal{S}_1 given in [24, Lemma 8.1] for the action of $\mathcal{G}_3^0(X_1)$ on $\mathcal{A}_2(X_1) \times \Gamma_2^+(X_1)$. (Recall $X_1 = Y \times [-1, 1]$.) We define a smooth vector field \tilde{Z} on $(\mathcal{A}_2(X_1) \times \Gamma_2^+(X_1)) \times \mathbf{S}$ by

$$\tilde{Z}(g^*u, \mathbf{s}) = g^{-1}\Psi(\mathbf{s}),$$

for $g \in \mathcal{G}_3^0(X_1), u \in \mathcal{S}_1$ and $\mathbf{s} \in S^\infty$. We extend \tilde{Z} to $(\mathcal{A}_{2,loc}(X) \times \Gamma_{2,loc}^+(X)) \times S^\infty$ by setting

$$\tilde{Z}(u, \mathbf{s}) = \tilde{Z}(u|_{X_1}, \mathbf{s}).$$

By the construction of Ψ , \tilde{Z} has no zeros.

Lemma 3.8. *The vector field \tilde{Z} is gauge equivariant.*

Proof. For $g = g_0 g_1$ with g_0 a based gauge and $g_1 \in S^1$, we have

$$\begin{aligned}\tilde{Z}(g^*(u, \mathbf{s})) &= \tilde{Z}(g^*u, g_1^{-1}\mathbf{s}) = g_0^{-1}\tilde{Z}(u, g_1^{-1}\mathbf{s}) \\ &= g_0^{-1}g_1^{-1}\tilde{Z}(u, \mathbf{s}) = g^*\tilde{Z}(u, \mathbf{s}).\end{aligned}$$

□

Now we replace Z_e by \tilde{Z} in the stable transition flow equation (3.1). The equation remains invariant under the action of $\mathcal{G}_{2,loc}$ and admits no reducible solution. Restricting to $S^{2n-1} \subset S^\infty$ we obtain the associated stablized moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^{2n-1}, S_{\alpha_+} \times S^{2n-1})$. Their direct limits as $n \rightarrow \infty$ are the total moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^\infty, S_{\alpha_+} \times S^\infty)$. Note that in the case $\mathbf{S} = S^\infty$, it suffices for our purpose to utilize the subspaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^{2n+1}, S_{\alpha_+} \times S^{2n+1})$ instead of the total spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^\infty, S_{\alpha_+} \times S^\infty)$.

Theorem 3.9. *The moduli spaces $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^{2n+1}, S_{\alpha_+} \times S^{2n+1})$ are compact Hausdorff spaces. Moreover, for generic b_0 , we have for all $\alpha_- \in \mathcal{R}_-, \alpha_+ \in \mathcal{R}_+$ and n*

- (1) $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^{2n-1}, S_{\alpha_+} \times S^{2n-1})$ has the structure of d -dimensional smooth oriented manifolds with corners, where $d = \mu_-(\alpha_-) - \mu_+(\alpha_+) + m_0 - \dim G_{\alpha_-} + 2n$.
- (2) This structure is compatible with the natural stratification of $\underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^{2n-1}, S_{\alpha_+} \times S^{2n-1})$.
- (3) The projections $\pi_\pm : \underline{\mathbf{M}}_T^0(S_{\alpha_-} \times S^{2n-1}, S_{\alpha_+} \times S^{2n-1}) \rightarrow S_{\alpha_\pm} \times S^{2n-1}$ are S^1 -equivariant smooth maps.

Passing to S^1 quotient, we obtain the quotient version of Theorem 3.9.

With these preparations, it is clear that we can carry over the above proof of Theorem 3.1 to the general case $\mathbf{S} = S^{2n-1}$ or $\mathbf{S} = S^\infty$.

4. DE RHAM VERSION

In this section we construct the de Rham version of equivariant Seiberg-Witten Floer homology and cohomology and prove that it is isomorphic to the singular version with real coefficients. We also construct the de Rham version of Bott-type Seiberg-Witten Floer homology and cohomology and prove that it is isomorphic to the singular version of Bott-type theory with real coefficients.

Equivariant Theory

As before, $\mathbf{S} = S^{2n-1}$ or $\mathbf{S} = S^\infty$. Choose an S^1 invariant Riemannian metric on each S_α and use the standard metric on S^{2n-1} . (Of course, there is no need for choosing a metric on the reducible S_α .) Then we obtain the adjoint d^* of the exterior differential d on $S_\alpha \times_{S^1} S^{2n-1}$. Set $\Omega^*(\mathcal{R}^0 \times_{S^1} S^{2n-1}) = \bigoplus_\alpha \Omega^*(S_\alpha \times_{S^1} S^{2n-1})$. We have natural inclusions $\Omega^*(\mathcal{R}^0 \times_{S^1} S^{2n-1}) \subset \Omega^*(\mathcal{R}^0 \times_{S^1} S^{2n+1})$ which commute with d and d^* . Using this fact we define $(\Omega^*(\mathcal{R}^0 \times_{S^1} \mathbf{S}), d, d^*)$ to be the direct limit of $(\Omega^*(\mathcal{R}^0 \times_{S^1} S^{2n-1}), d, d^*)$ as n approaches infinity.

Now we introduce our equivariant de Rham complex $C_{equ, deR}^*$,

$$C_{equ, deR}^k \oplus \Omega^i(S_\alpha \times_{S^1} \mathbf{S})$$

We define $C_*^{equ,deR}$ to be the same as $C_{equ,deR}^*$. Next we define the desired coboundary operator. We first consider the case $\mathbf{S} = S^{2n-1}$. For each pair $\alpha, \beta \in \mathcal{R}$ with $\mu(\alpha) > \mu(\beta)$ we define a coboundary operator $\partial_{\alpha,\beta}^*$ as follows. If the moduli space $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} S^{2n-1}, S_\beta \times_{S^1} S^{2n-1})$ is empty, we define $\partial_{\alpha,\beta}^*$ to be the zero operator. If it is nonempty, we set $\partial_{\alpha,\beta}^* \omega = 0$ for $\omega \notin \Omega^*(S_\beta \times_{S^1} S^{2n-1})$. For $\omega \in \Omega^j(S_\beta \times_{S^1} S^{2n-1})$, we set

$$\partial_{\alpha,\beta}^* \omega = (-1)^j (\pi_-)_* \pi_+^* \omega,$$

where $(\pi_-)_*$ denotes the fiber integration on $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} S^{2n-1}, S_\beta \times_{S^1} S^{2n-1})$ with respect to the fibration π_- . (For the simple definition of fiber integration see e.g. [5].) We define the coboundary operator of the equivariant de Rham complex to be

$$\partial_{equ,deR}^* = d + \sum_{\mu(\alpha) > \mu(\beta)} \partial_{\alpha,\beta}^*.$$

Following the arguments in [5] we deduce $(\partial_{equ,deR}^*)^2 = 0$.

Since $\Omega^j(S_i \times_{S^1} S^\infty)$ is the direct limit of $\Omega^j(S_i \times_{S^1} S^{2n-1})$, the above definition of $\partial_{equ,deR}^*$ immediately extends to cover the case $\mathbf{S} = S^\infty$. Namely each $\omega \in \Omega^j(S_\beta \times_{S^1} S^\infty)$ can be viewed as in $\Omega^j(S_\beta \times_{S^1} S^{2n-1})$ for some n , and hence the above definition of the coboundary operator can be applied. On the other hand, we can take the direct limit of the fiber integration $(\pi_-)_*$ on $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} S^{2n-1}, S_\beta \times_{S^1} S^{2n-1})$ to obtain a fiber integration $(\pi_-)_*$ on $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} S^\infty, S_\beta \times_{S^1} S^\infty)$. Then we carry over the above definition of $\partial_{equ,deR}^*$ word for word to $\mathbf{S} = S^\infty$.

Definition 4.1. *The de Rham version of equivariant Seiberg-Witten Floer cohomology $FH_{equ,deR}^{SW*}$ is defined to be the cohomology of the complex*

$$(C_{equ,deR}^*, \partial_{equ,deR}^*).$$

Next we construct the de Rham version homology theory, using the complex $C_*^{equ,deR}$. The associated boundary operator is defined as follows. For $\alpha, \beta \in \mathcal{R}$ with $\mu(\alpha) > \mu(\beta)$ and $\omega \in \Omega^j(S_\alpha \times_{S^1} \mathbf{S})$, we set

$$\partial_{\alpha,\beta} \omega = (-1)^j (\pi_+)_* \pi_-^* \omega.$$

(As before, this operator is defined to be the zero operator for other ω or for empty $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} \mathbf{S}, S_\beta \times_{S^1} \mathbf{S})$.)

We define

$$\partial_{equ,deR} = d^* + \sum_{\mu(\alpha) > \mu(\beta)} \partial_{\alpha,\beta}.$$

We have $\partial_{equ,deR}^2 = 0$.

Definition 4.2. *The de Rham version of equivariant Seiberg-Witten Floer homology*

$$FH_{equ,deR}^{SW}$$

is defined to be the homology of the complex $(C_^{equ,deR}, \partial_{equ,deR})$.*

We have the equivariant de Rham index filtration $\mathcal{F}_{equ,deR}^*$ and its dual filtration $\mathcal{F}_*^{equ,deR}$,

$$\mathcal{F}_{equ,deR}^k = \bigoplus_{j \geq k} \Omega^*(S_j \times_{S^1} \mathbf{S}),$$

$$\mathcal{F}_k^{equ,deR} = \bigoplus_{j \leq k} \Omega^*(S_j \times_{S^1} \mathbf{S}).$$

Theorem 4.3. *The equivariant de Rham index filtrations induce spectral sequences $E_{**}^*(equ, deR)$ and $E_{**}^*(equ, deR; dual)$ such that the former converges to*

$$FH_{equ, deR}^{SW*},$$

and the latter converges to

$$FH_{equ, deR*}^{SW}.$$

Moreover, we have

$$(4.1) \quad E_{ij}^1(equ, deR) \cong H_{deR}^j(S_i \times_{S^1} \mathbf{S}),$$

$$(4.2) \quad E_{ij}^1(equ, deR; dual) \cong H_j^{deR}(S_i \times_{S^1} \mathbf{S}).$$

This is similar to Theorem 2.2.

Theorem 4.3. *The de Rham version of equivariant Seiberg-Witten Floer homology and cohomology are diffeomorphism invariants. Indeed, we have $(FH_{equ, deR}^{SW})^j \cong (FH_{equ; \mathbb{R}}^{SW})^j, (FH_{equ, deR}^{SW})_j \cong (FH_{equ; \mathbb{R}}^{SW})_j$.*

Proof. We present the arguments for cohomologies. Homologies can be handled in a similar way. We have a natural cochain map $F : C_{equ, deR}^* \rightarrow C_{equ; \mathbb{R}}^*$, which is defined in terms of integrating forms along generalized cubical singular chains. The fact that this is indeed a cochain map follows quite easily from the construction of the coboundary operators $\partial_{equ}^*, \partial_{equ, deR}^*$ and Stokes' theorem.

It is easy to see that F preserves the index filtrations. The induced homomorphism $F^* : GC_{equ, deR}^* \rightarrow GC_{equ; \mathbb{R}}^*$ consists of the cochain homomorphisms from $(\Omega^*(S_j \times_{S^1} \mathbf{S}), d)$ to $(C^*(S_j \times_{S^1} \mathbf{S}), \pm \partial_O^*)$ defined in terms of integration. Here $\pm \partial_O^*$ denotes the singular coboundary operator ∂_O^* modified by signs, with sign convention $\pm = (-1)^{i+j}$ on $C^i(S_j \times_{S^1} \mathbf{S})$, cf. [24, Section 7]. It follows that the induced cochain homomorphism between the E^1 terms $E_{**}^1(equ, deR)$ and $E_{**}^1(equ, dual; \mathbb{R})$ is an isomorphism. Since we are using the real coefficients, this implies, on account of the convergence of the involved spectral sequences, the desired isomorphism. \square

Bott-type Theory

The construction is parallel to the one above. We have in the first place the de Rham version Bott-type complex

$$C_{Bott, deR}^k = \oplus_{i+j=k} \Omega^j(S_i).$$

The coboundary operator $\partial_{Bott, deR}^*$ for this cochain complex is defined analogously to $\partial_{equ, deR}^*$, where we use $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ instead of $\underline{\mathbf{M}}_T^0(S_\alpha \times_{S^1} \mathbf{S}, S_\beta \times_{S^1} \mathbf{S})$.

Definition 4.4. We define the de Rham version Bott-type Seiberg-Witten Floer cohomology $FH_{Bott, deR}^{SW*}$ to be the cohomology of $(C_{Bott, deR}^*, \partial_{Bott, deR}^*)$. We also have the de Rham version Bott-type homology $FH_{Bott, deR*}^{SW}$.

We omit the statement of the spectral sequence theorem here. In analogy to Theorem 4.3, we have

Theorem 4.5. *We have natural isomorphisms*

$$FH_{Bott, deR}^{SW*} \cong FH_{Bott; \mathbb{R}}^{SW*}, FH_{Bott, deR*}^{SW} \cong FH_{Bott; \mathbb{R}*}^{SW}.$$

5. CARTAN VERSION

In this section we give a brief account of the construction of the Cartan version of equivariant Seiberg-Witten Floer homology and cohomology and prove that it is isomorphic to the de Rham version. The construction is in spirit similar to that in [4]. Such a construction first appeared in the paper [19].

We follow the presentation of the Cartan theory given in [6]. For a compact Lie group G acting smoothly on a manifold N , we have the following space of G -equivariant differential forms on G

$$\Omega_G^*(N) = (\Omega^*(N) \otimes \mathbb{C}[\mathfrak{g}^*])^G,$$

which is the subalgebra of G -equivariant elements in the algebra $\Omega^*(N) \otimes \mathbb{C}[\mathfrak{g}]$. Here $\mathbb{C}[\mathfrak{g}^*]$ denotes the complex polynomial algebra on the dual Lie algebra \mathfrak{g}^* of G (i.e. the algebra of complex valued polynomial functions on \mathfrak{g}), G acts on the ordinary form part by pullback, and on the Lie algebra part by the adjoint representation. The algebra $\Omega^*(N) \otimes \mathbb{C}[\mathfrak{g}^*]$ has a \mathbb{Z} -grading defined by

$$\deg(w \otimes z) = \deg(w) + 2 \deg(z),$$

for $w \in \Omega^*(N)$ and $z \in \mathbb{C}[\mathfrak{g}^*]$. There is a natural differential d_G on $\Omega_G^*(N)$

$$(d_G \alpha)(Z) = d(\alpha(Z)) - \iota(Z)(\alpha(Z)),$$

for $\alpha \in \Omega_G^*(N)$, where $\iota(Z)$ denotes contraction by the vector field Z_N induced by an element $Z \in \mathfrak{g}$ via the action of G . One easily verifies $d_G^2 = 0$.

Now consider our set-up of the Seiberg-Witten theory on the 3-manifold Y as before. We introduce the complex C_{Cartan}^* ,

$$C_{Cartan}^k = \oplus_{i+j=k} \Omega_{S^1}^j(S_i).$$

For each pair α, β with $\mu(\alpha) > \mu(\beta)$, we define an operator $\partial_{\alpha, \beta}^* : C_{Cartan}^k \rightarrow C_{Cartan}^{k+1}$ as follows. If the moduli space $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ is empty, we define $\partial_{\alpha, \beta}^*$ to be the zero operator. If it is nonempty, we set $\partial_{\alpha, \beta}^* \omega = 0$ for $\omega \notin \Omega_{S^1}^*(S_\beta)$. For $\omega \in \Omega_{S^1}^j(S_\beta)$, we set

$$\partial_{\alpha, \beta}^* \omega = (-1)^j (\pi_-)_* \pi_+^* \omega.$$

The gauge equivariance of the projections π_\pm of $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ ensures that this indeed produces an equivariant form. We define the coboundary operator of the Cartan complex to be

$$\partial_{Cartan}^* = d_{S^1} + \sum_{\mu(\alpha) > \mu(\beta)} \partial_{\alpha, \beta}^*.$$

Following the arguments in [5] we deduce $(\partial_{Cartan}^*)^2 = 0$.

Definition 5.1. *The Cartan version of equivariant Seiberg-Witten Floer cohomology FH_{Cartan}^{SW*} is defined to be the cohomology of the complex $(C_{Cartan}^*, \partial_{Cartan}^*)$.*

In analogy to the dual de Rham construction, we also have a dual Cartan construction which produces the homology FH_{Cartan}^{SW} . We omit the simple details.

Theorem 5.2. *The Cartan version of equivariant Seiberg-Witten Floer cohomology and homology are diffeomorphism invariants. Indeed, we have $FH_{Cartan}^{SW*} \cong FH_{equ,deR;S^\infty}^{SW*}, FH_{Cartan*}^{SW} \cong FH_{equ,deR;S^\infty*}^{SW}$.*

Proof. We follow the arguments in [2] and [5]. Consider the principal S^1 -bundle $\pi : S^{2n-1} \rightarrow \mathbb{C}P^n$. We choose the natural S^1 -connection on this bundle which is induced by the Euclidean metric structure of \mathbb{R}^{2n} . Denote its curvature form by $\Omega(n)$. Using the inclusions $S^{2n-1} \subset S^{2n+1}$ we then obtain the direct limit of $\Omega(n)$ as n approaches infinity, which will be denoted by Ω . Note that Ω belongs to $\Omega^2(S^\infty)$.

Now consider the principal S^1 -bundles $\pi : S_\alpha \times S^\infty \rightarrow S_\alpha \times_{S^1} S^\infty$ with $\mathbb{C}P^\infty$ as fiber. We define a natural homomorphism

$$\tilde{F} : \Omega_{S^1}^*(S_\alpha) \rightarrow \Omega_{basic}^*(S_\alpha \times S^\infty),$$

where basic forms mean forms which are horizontal and S^1 -invariant. For generators $\omega \otimes u \in \Omega_{S^1}^*(S_\alpha)$, we set

$$\tilde{F}(\omega \otimes u) = \omega \wedge u(\Omega).$$

Composing \tilde{F} with the inverse of the natural isomorphism $\pi^* : \Omega^*(S_\alpha \times_{S^1} S^\infty) \rightarrow \Omega_{basic}^*(S_\alpha \times S^\infty)$ we then obtain a natural cochain homomorphism

$$F_\alpha : (\Omega_{S^1}^*(S_\alpha), d_{S^1}) \rightarrow (\Omega^*(S_\alpha \times_{S^1} S^\infty), d).$$

By [2], this homomorphism induces an isomorphism F_α^* on cohomologies.

Putting all F_α together we obtain $F : C_{Cartan}^* \rightarrow C_{equ,deR}^*$ with $F : C_{Cartan}^k \rightarrow C_{equ,deR}^k$. By the constructions of the involved coboundary operators it is easy to see that F is a cochain map.

By the construction, F preserves the index filtrations. The induced homomorphism $F^* : GC_{Cartan}^* \rightarrow GC_{equ,deR}^*$ precisely consists of the cochain homomorphisms $F_\alpha : (\Omega_{S^1}^*(S_\alpha), d_{S^1}) \rightarrow (\Omega^*(S_\alpha \times_{S^1} S), d)$. It follows that the induced cochain homomorphism between the E^1 terms $E_{**}^1(Cartan)$ and $E_{**}^1(equ, deR)$ is an isomorphism. On account of the convergence of the involved spectral sequences, this implies the desired isomorphism. \square

6. EXPONENTIAL CONVERGENCE

In this section we first prove the exponential asymptotics result [24, Propostion 4.2]. We restate it in a strengthened form below.

Propostion 6.1. *Consider a compact set $K_{h,\lambda,\pi}$ of triples (h, λ, π) , where h denotes a metric on Y and (λ, π) a pair of Y -generic parameters corresponding to h . Then there are positive constants δ_0 and E_0 depending only on $K_{h,\lambda,\pi}$ with the following properties. Let $(h, \lambda, \pi) \in K_{h,\lambda,\pi}$. Let $u = (A, \Phi) = (a, \phi)$ be a temporal trajectory of local $(2, 2)$ -Sobolev class and finite energy. Then there exist a gauge $g \in \mathcal{G}_2(Y)$ and two smooth solutions u_-, u_+ of (1.1) such that $\tilde{u} = g^*u$ is smooth and the following holds. For all l ,*

$$\|\tilde{u} - u_+\|_{l, \delta_0; Y \times [T_1, \infty)} \leq C(l),$$

$$\|\tilde{u} - u_-\|_{l, \delta_0; Y \times [T_1, \infty)} \leq C(l)$$

where $C(l)$ depends only on $K_{h,\lambda,\pi}$, l and an upper bound of the energy of u , and $T_1 > 0, T_2 < 0$ satisfy $E(u, Y \times [T_1, \infty)), E(u, Y \times (-\infty, T_2]) \leq E_0$. (Recall that $E(u, \Omega)$ means the energy of u on the domain Ω .)

If u_- and u_+ are not gauge equivalent, then

$$E(u) \geq E_0.$$

Furthermore, the above weighted Sobolev estimates hold for any smooth temporal trajectory \tilde{u} provided that all the cited conditions hold and in addition its values at T and T' are under smooth control, with the constants $C(l)$ also depending on the said control.

Note that under the rational homology sphere assumption, the energy $E(u)$ vanishes when u_- and u_+ are gauge equivalent. This follows from [24, Lemma 2.9].

We fix a compact set $K_{h,\lambda,\pi}$. In the remainder of this section, all the results are under this condition, and all the constants in the various estimates depend on $K_{h,\lambda,\pi}$.

Let $u = (A, \Phi)$ satisfy the conditions in Proposition 6.1. We handle convergence at $+\infty$. The situation of $-\infty$ is similar.

For an arbitrary r , consider the cylinder $X_{r,r+2} = Y \times [r, r+2]$. We apply [24, Lemma B.1] to produce the Columb form u^r of u over this cylinder. By the proof of [24, Lemma 8.6], we have

$$\begin{aligned} \int_{X_{r,r+2}} |F_A|^2 &\leq 2 \int_{X_{r,r+2}} |F_A + \langle e_i \cdot \phi, \phi \rangle * e^i|^2 + 6 \int_{X_{r,r+2}} |\phi|^4 \\ (6.1) \quad &\leq 4 \int_{X_{r,r+2}} |*F_A + \langle e_i \cdot \phi, \phi \rangle * e^i|^2 + 4 \int_{X_{r,r+2}} \left| \frac{\partial a}{\partial t} - d_Y f \right|^2 \\ &+ 6 \int_{X_{r,r+2}} |\phi|^4 \leq C(1 + E(r)). \end{aligned}$$

By [24, Lemma B.1] we then obtain a uniform estimate for the $L^{1,2}$ norm of $A - A_0$ over $X_{r,r+2}$, where $A_0 = a_0$ is a reference connection, with a_0 being a fixed smooth connection over Y . Now $u^r = (A, \Phi) = (a + f dt, \phi)$ satisfies the following elliptic system on $Y \times (r, r+2)$ (cf. [24, (2.10)])

$$(6.2) \quad \begin{cases} F_A^+ = \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j + \nabla H(a) \wedge dt + *(\nabla H(a) \wedge dt), \\ d^*(A - A_0) = 0, \\ D_A \Phi = -\lambda \frac{d}{dt} \cdot \Phi. \end{cases}$$

Using this system, the estimate (6.1), [24, Lemma 8.5] and basic elliptic theory, we then derive uniform (independent of r) interior smooth estimates for u^r . For an arbitrary sequence $r_k \rightarrow \infty$, we obtain a subsequence, which we still denote by r_k , and an $L^{1,2}$ weak limit $u_\infty = (a_\infty + f_\infty dt, \phi_\infty)$ of $u_k := u^{r_k}(\cdot, \cdot - r_k)$, such that $f_\infty \in L_0^{1,2}(Y \times [0, 2])$, i.e. f_∞ has zero boundary trace. (The zero boundary trace condition follows from the Columb boundary condition of [24, Lemma B.1].) Moreover, u_k converges to u_∞ smoothly in the interior. By the finite energy condition, u_∞ satisfies the following equations (we omit the subscripts (λ, H) in the perturbed Seiberg-Witten equation):

$$(6.2) \quad \begin{cases} F_A^+ = \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j + \nabla H(a) \wedge dt + *(\nabla H(a) \wedge dt), \\ d^*(A - A_0) = 0, \\ D_A \Phi = -\lambda \frac{d}{dt} \cdot \Phi. \end{cases}$$

for all $t \in (0, 2)$, and

$$\frac{\partial a_\infty}{\partial t} - d_Y f_\infty = 0, \frac{\partial f_\infty}{\partial t} - d_Y^*(a_\infty - a_0) = 0, \frac{\partial \phi_\infty}{\partial t} + f_\infty \phi_\infty = 0,$$

on $Y \times (0, 2)$. It follows that f_∞ is harmonic and hence $f_\infty \equiv 0$. Consequently, u_∞ has no t -component and is t -independent. Moreover, we have

$$(6.4) \quad d_Y^*(a_\infty - a_0) = 0.$$

We write $u_k = (a_k + f_k dt, \phi_k)$ as usual. Applying the gauge fixing lemma [24, Lemma 3.2] we convert $(a_k(\cdot, t), \phi_k(\cdot, t))$ for large k and t strickly inside $(0, 2)$ into satisfying the gauge fixing condition

$$(6.5) \quad d_Y^*(a - a_\infty) + \text{Im} \langle \phi_\infty, \phi \rangle = 0.$$

By [24, Lemma 3.2] and basic elliptic estimates we can control the involved gauges, and hence derive that the new (a_k, ϕ_k) still converges to u_∞ (we retain the notation (a_k, ϕ_k) for convenience).

Lemma 6.2. *For large k we have*

$$(6.6) \quad \|\mathbf{sw}(a_k, \phi_k)\|_{L^2}^2 \geq c \|(b_k, \psi_k)\|_{L^{1,2}}^2$$

for a positive constant c , where $b_k = a_k - a_\infty, \psi_k = \phi_k - \phi_\infty$.

Proof. We first express

$$\mathbf{sw}(a_k, \phi_k) = \mathbf{sw}(a_k, \phi_k) - \mathbf{sw}(a_\infty, \phi_\infty)$$

in terms of $L_k(b_k, \psi_k)$, where L_k denotes the linearized Seiberg-Witten operator $d\mathbf{sw}(a_\infty + s(a_k - a_\infty), \phi_\infty + s(\phi_k - \phi_\infty))$ for $s \in [0, 1]$. Then we write $L_k = d\mathbf{sw}(a_\infty, \phi_\infty) + (L_k - d\mathbf{sw}(a_\infty, \phi_\infty))$. By the convergence of u_k to u_∞ , the term involving $L_k - d\mathbf{sw}(a_\infty, \phi_\infty)$ approaches zero as k goes to infinity. On the other hand, by the genericity assumption on the perturbation parameters we have the estimate

$$(6.7) \quad \|d\mathbf{sw}(a_\infty, \phi_\infty)(b', \psi')\|_{L^2} \geq c \|(b', \psi')\|_{L^{1,2}}$$

for a positive constant c and all (b', ψ') satisfying the gauge condition (6.5). This is obvious in the case that u_∞ is irreducible, for in that case $\ker d\mathbf{sw}(u_\infty)|_{G_{Y, u_\infty}^*}$ is obviously trivial under the genericity assumption. (See [24, Section 3] for the infinitesimal gauge action operator G_{Y, u_∞} and other relevant materials.) If u_∞ is reducible, then we have (see [24] for the definition of \mathcal{D}_1)

$$\|\mathcal{D}_1|_{(a_\infty, \phi_\infty, c)}(b, \psi, f)\|_{L^2} \geq c \|(b, \psi, f)\|_{L^{1,2}},$$

for all $(b, \psi, f) \in \ker d_1^* \oplus \Gamma_1(S) \oplus \Omega_1^0(Y)$ which are orthogonal to the kernel of \mathcal{D}_1 . But this kernel obviously contains $\{0\} \oplus \{0\} \oplus i\mathbb{R}$. Since \mathcal{D}_1 is surjective by the genericity assumption and $\text{ind } \mathcal{D}_1 = 1$, $\{0\} \oplus \{0\} \oplus \mathbb{R}$ must be the entire kernel. It follows that all elements of the form $(b', \psi', 0)$ in $\ker d_1^* \oplus \Gamma_1(S) \oplus \Omega_1^0(Y)$ are orthogonal to $\ker \mathcal{D}_1$. Finally, we observe $\mathcal{D}_1(b', \psi', 0) = d\mathbf{sw}(a_\infty, \phi_\infty)(b', \psi')$. \square

It is easy to extend the above results to a more general setting as follows

Lemma 6.3. *For each $\delta > 0$ there is a positive number $E_0 = E_0(\delta)$ with the following property. Let u be a solution of (1.2) over $Y \times [T_1, T_2]$ with $1 \leq T_2 - T_1 \leq 2$. If $E(u) \leq E_0(\delta)$, then there is a solution $u_\infty = (a_\infty, \phi_\infty)$ of (1.1) such that $d^*(a_\infty - a_0) = 0$ and*

$$\|\tilde{u} - u_\infty\|_{C^2(Y \times [T_1+1/2, T_2-1/2])} \leq \delta,$$

$$\|\tilde{u} - u_\infty\|_{C^j(Y \times [T_1+1/2, T_2-1/2])} \leq \epsilon(\delta, j),$$

where \tilde{u} is obtained from u by a suitable gauge transformation (u_∞ is the same for all j), and the functions $\epsilon(\delta, j)$ approach zero as $\delta \rightarrow 0$. If we write $\tilde{u} = (\tilde{a} + \tilde{f}dt, \tilde{p})$, then (\tilde{a}, \tilde{p}) satisfies the gauge condition (6.5). Moreover, there holds

$$\|\mathbf{sw}(\tilde{u})\|_{L^2} \geq c\|\tilde{u} - u_\infty\|_{L^{1,2}(Y)}$$

for a positive constant c , at each $t \in [T_1 + 1/2, T_2 - 1/2]$.

For different sequences u_k as above, we may a priori get different limits u_∞ . However, by the monotonicity of \mathbf{cs} , the limit value $\mathbf{cs}(\infty) = \mathbf{cs}(u_\infty)$ is independent of the choice of the sequence. Indeed, it is equal to $\lim_{t \rightarrow \infty} \mathbf{cs}(a, \phi)$.

Now we proceed to derive energy decay. We set $J(t) = \mathbf{cs}(t) - \mathbf{cs}(\infty) \equiv \mathbf{cs}((a(t), \phi(t)) - \mathbf{cs}(\infty))$. Working in temporal gauge we deduce

$$(6.8) \quad \frac{\partial J}{\partial t} = \frac{\partial \mathbf{cs}}{\partial t} = \nabla \mathbf{cs} \cdot \frac{\partial u}{\partial t} = -\|\mathbf{sw}(a, \phi)\|_{L^2}^2.$$

On the other hand, we have by [24, Lemma 2.5]

$$(6.9) \quad J(t) = \frac{1}{2}E(u, Y \times [t, \infty)).$$

By gauge invariance, the formulas (6.8) and (6.9) hold in any gauge. By the above convergence argument, the distance from the suitable gauge form of $(a(t), \phi(t)) := (a(\cdot, t), \phi(\cdot, t))$ to some u_∞ goes to zero as t goes to infinity. For each large t , we use the corresponding u_∞ and the associated gauge. By Lemma 6.2 or Lemma 6.3 we deduce

$$(6.10) \quad \|\mathbf{sw}(a(t), \phi(t))\|_{L^2} \geq c\|(b, \psi)\|_{L^{1,2}}$$

with $(b, \psi) = (a(t) - a_\infty, \phi(t) - \phi_\infty)$. On the other hand, we have on account of a priori estimates in the Columb gauge

$$\|\mathbf{sw}(s(a(t), \phi(t)) + (1-s)(a_\infty, \phi_\infty))(b', \psi')\|_{L^2} \leq C\|(b', \psi')\|_{L^{1,2}}$$

with $s \in [0, 1]$. Here and in the sequel, C denotes a positive constant whose value depends on each context. Consequently,

$$(6.11) \quad \begin{aligned} J(t) &= \mathbf{cs}(t) - \mathbf{cs}(\infty) \\ &= \int_0^1 \mathbf{sw}(s(a(t), \phi(t)) + (1-s)(a_\infty, \phi_\infty))(b(t), \psi(t)) ds \\ &\leq C\|(b(t), \phi(t))\|_{L^{1,2}}. \end{aligned}$$

Combining (6.8), (6.10) and (6.11) we then infer

Lemma 6.4. *There are positive constants $E_0 > 0$ and c such that*

$$(6.12) \quad J(t) \leq -cJ'(t)$$

whenever $E(u, Y \times [t-1, \infty)) \leq E_0$. Consequently,

$$(6.13) \quad J(t) \leq J(t_0)e^{-c(t-t_0)}$$

for $t \geq t_0$, provided that $E(u, Y \times [t_0-1, \infty)) \leq E_0$.

Next we derive an estimate of $\mathbf{sw}(a, \phi)$ in terms of energy.

Lemma 6.5. *There holds*

$$(6.14) \quad \|\mathbf{sw}(a, \phi)\|_{L^\infty(Y \times \{t\})} \leq CE(u, Y \times [t-1, t+1])^{1/2}$$

for $u = (a + fdt, \phi)$ in any gauge form.

Proof. Fix t_0 . We use the Columb form u_C of u over $Y \times [t_0-1, t_0+1]$. Write $u_C = (a + fdt, \Phi)$. Using the temporal gauge $g = \exp(\int_{t_0-1}^t f)$ we convert u into a temporal form $\tilde{u} = (\tilde{a}, \tilde{\phi})$, for which we have smooth control over $Y \times [k-1/2, k+1/2]$. We set

$$v = (\tilde{b}, \tilde{\psi}) = \mathbf{sw}(\tilde{a}, \tilde{\phi}).$$

There holds

$$(6.15) \quad \begin{cases} \frac{\partial v}{\partial t} = d\mathbf{sw}(\tilde{a}, \tilde{\phi})(v), \\ G_{Y, \tilde{u}}^* v = 0. \end{cases}$$

Setting $\tilde{f} = 0$, $\tilde{A} = \tilde{b} + \tilde{f}dt$, and $\tilde{\Psi}(\cdot, t) = \tilde{\psi}$ we deduce

$$(6.16) \quad \begin{cases} d\mathbf{SW}(\tilde{u})(\tilde{A}, \tilde{\Psi}) = 0, \\ G_{X, \tilde{u}}^*(\tilde{A}, \tilde{\Psi}) = 0. \end{cases}$$

This elliptic system has been employed in [24, Section 4]. Its coefficients are given in terms of \tilde{u} . On the other hand, by gauge invariance we have

$$\|(\tilde{A}, \tilde{\Psi})\|_{L^2(Y \times [k-1, k+1])} \leq CE(u, Y \times [k-1, k+1]).$$

By basic elliptic theory we then obtain higher order estimates for $(\tilde{A}, \tilde{\Psi})$. In particular, we obtain

$$\|v\|_{L^\infty_{Y \times \{t\}}} \leq CE(u, Y \times [t-1, t+1])^{1/2}.$$

□

Proof of Proposition 6.1.

Using the Columb gauges and a patching argument, we can arrange u to be smooth everywhere. Indeed, we can start converting u into the Columb gauge on $Y \times [-2, 2]$. We extend the involved gauge to the entire $Y \times \mathbb{R}$ by interpolating

with the identity. Next we convert u on $Y \times [1, 5]$ and $Y \times [-5, -1]$ with a Columb gauge g . We interpolate g such that it becomes the identity over $Y \times [1, 3/2]$ and $Y \times [-1, -3/2]$. Then the resulting u will be smooth over $Y \times (-5, 5)$. Moreover, we have smooth estimates for it in the interior. Arguing like this, we can convert u step by step. The final u we obtain is no longer in Columb gauge everywhere because of interpolation of the gauges. But it is smooth and we have uniform smooth estimates for it.

We denote this smooth form of u by u_S . Now we convert u_S into a temporal form by using the temporal gauge $g = \exp(\int_{T_0}^t f_S)$ with T_0 satisfying $E(Y \times [T_0, \infty)) \leq E_0$. We denote it by u_P . It is smooth and its value at T_0 is under smooth control.

By Lemma 6.4 and Lemma 6.5 we deduce

$$(6.17) \quad \|\mathbf{sw}(u_P)\|_{L^\infty} \leq Ce^{-c(t-T_0)}.$$

By integration, we first derive exponential L^∞ convergence for u_P because we have

$$\frac{\partial u_P}{\partial t} = \mathbf{sw}(u_P).$$

We can apply the elliptic system (6.16) to u_P and $\mathbf{sw}(u_P)$. By basic elliptic estimates we then deduce $L^{1,l}$ decay estimate for $\mathbf{sw}(u_P)$ for any l . Integrating, this implies in turn exponential $L^{1,l}$ convergence for u_P . Differentiating the system (6.16) and iterating we then deduce higher order decay estimates for $\mathbf{sw}(u_P)$ and high order exponential convergence for u_P . The claimed estimates in exponentially weighted Sobolev norms follow readily.

It is clear that these arguments apply to any temporal form \tilde{u} of u as long as the value of \tilde{u} at T_0 is under smooth control. \square

The following generalization of Lemma 6.4 is very important for us.

Lemma 6.6. *There are positive numbers E_0 and c with the following properties. Let u be a solution of (1.2) over $Y \times [T_1 - 1, T_2 + 1]$ such that $T_2 - T_1 \geq 2$ (this assumption is purely for the purpose of concise formulation) and $E(u, Y \times [T_1 - 1, T_2 + 1]) \leq E_0$. Then at least one of the following three cases occurs.*

Case 1

$$E(u, Y \times [t, T_2]) \leq E(u)e^{-c(t-T_1)}$$

for all $t \in [T_1, T_2]$.

Case 2

$$E(u, Y \times [T_1, t]) \leq E(u)e^{-c(T_2-t)}$$

for all $t \in [T_1, T_2]$.

Case 3 There is some $T_0 \in [T_1, T_2]$ such that

$$E(u, Y \times [t, T_0]) \leq E(u)e^{-c(t-T_1)}$$

for all $t \in [T_1, T_0]$ and

$$E(u, Y \times [T_0, t]) \leq E(u)e^{-c(T_2-t)}$$

for all $t \in [T_0, T_2]$.

Note that Case 1 and Case 2 can be considered as special cases of Case 3.

Proof. Applying Lemma 6.3 to each interval $[t_1, t_2]$ in $[T_1 - 1, T_2 + 1]$ of length 1, we obtain a limit u_∞ . The limits may depend on the interval, but the Chern-Simons value $\mathbf{cs}(u_\infty)$ doesn't. Indeed, by the Y -genericity assumption, there are only finitely many elements in \mathcal{R} , the moduli space of gauge classes of Seiberg-Witten points. We choose E_0 to be smaller than

$$\frac{1}{4} \min\{|\mathbf{cs}(\alpha) - \mathbf{cs}(\beta)| : \alpha, \beta \in \mathcal{R}, \alpha \neq \beta\}.$$

We also choose E_0 so small such that the estimates in Lemma 6.3 imply

$$|\mathbf{cs}(u(\cdot, t)) - \mathbf{cs}(u_\infty)| < \frac{1}{4} \min\{|\mathbf{cs}(\alpha) - \mathbf{cs}(\beta)| : \alpha, \beta \in \mathcal{R}, \alpha \neq \beta\}.$$

Employing the equation [24, (2.18)] relating the energy to the Chern-Simons functional, we then conclude that $\mathbf{cs}(u_\infty)$ is independent of u_∞ . We denote it by $\mathbf{cs}(\infty)$.

There are three cases to consider. We set $\mathbf{cs}(t) = \mathbf{cs}(a(\cdot, t), \phi(\cdot, t))$ as before.

Case 1 $\mathbf{cs}(\infty) < \mathbf{cs}(T_2)$.

In this case, we consider the function $J(t) = \frac{1}{2}E(u, Y \times [t, T_2]) + \mathbf{cs}(T_2) - \mathbf{cs}(\infty) = \mathbf{cs}(t) - \mathbf{cs}(\infty)$ and argue as in the proof of Lemma 6.4. We obtain decay of J . Since $\mathbf{cs}(T_2) - \mathbf{cs}(\infty) > 0$, this implies decay of the energy and leads to Case 1 in the lemma.

Case 2 $\mathbf{cs}(\infty) > \mathbf{cs}(T_1)$.

This is similar to Case 1, with the roles of T_1 and T_2 reversed. It corresponds to the negative time infinity version of Lemma 6.4 and leads to Case 2 of the lemma.

Case 3 $\mathbf{cs}(T_1) \geq \mathbf{cs}(\infty) \geq \mathbf{cs}(T_2)$.

In this case, there is a time $T_0 \in [T_1, T_2]$ such that $\mathbf{cs}(\infty) = \mathbf{cs}(T_0)$. We consider $J(t) = \mathbf{cs}(t) - \mathbf{cs}(T_0)$ for $t \in [T_1, T_0]$ and $J(t) = \mathbf{cs}(T_0) - \mathbf{cs}(t)$ for $t \in [T_0, T_2]$, and arrive at Case 3 of the lemma. \square

A simple consequence of Lemma 6.6 is the following result.

Lemma 6.7. *Assume the same as in Lemma 6.6. Consider the local energy of u*

$$E_u(t) \equiv E(u, [t - 1/2, t + 1/2]).$$

At least one of the following three cases occurs.

Case 1

$$E_u(t) \leq E(u)e^{-c(t-T_1-1/2)}$$

for all $t \in [T_1 + 1, T_2 - 1]$.

Case 2

$$E_u(t) \leq E(u)e^{-c(T_2-t-1/2)}$$

for all $t \in [T_1 + 1, T_2 - 1]$.

Case 3

$$E_u(t) \leq 2E(u)e^{-c(t-T_1-1/2)}$$

for all $t \in [T_1 + 1, T_0]$ and

$$E_u(t) \leq 2E(u)e^{-c(T_2-t-1/2)}$$

for all $t \in [T_0, T_2 - 1]$, where

$$T_0 = (T_1 + T_2)/2.$$

Next we derive a uniform pointwise estimate

Lemma 6.8. *Let u be a temporal trajectory satisfying an energy bound such that its value at some time t_0 is under smooth control. Then we have uniform pointwise smooth estimates for u . If we only assume an energy bound for u , we can find a gauge $g \in \mathcal{G}_3(Y)$ such that the value of g^*u at $t = 0$ is under smooth control in dependence on the energy bound. Consequently, we have smooth estimates for g^*u .*

Proof. The last statement follows from the arguments in the proof of Proposition 6.1. Assume $E(u) \leq C$ for some C and smooth control over the value of u at t_0 . For convenience, assume $t_0 = 0$. (Our arguments work for any t_0 .) We write \mathbb{R} as a union of m intervals $I_j = [t_j, t_{j+1}]$, $t_j < t_{j+1}$, with $m \leq 2E(u)/E_0$, such that for each j , either both $t_{j+1} - t_j > 1$ and

$$E(u, [t_j - 1, t_{j+1} + 1]) \leq E_0$$

hold, or $t_{j+1} - t_j = 1$. Here, we adopt the convention that e.g. $[t_j, t_{j+1}]$ means $(-\infty, t_{j+1}]$ if $t_j = -\infty$. To derive the desired estimates, we start with the interval I_{j_0} containing 0. If $E(u, I_{j_0}) \leq E_0$, we apply Lemma 6.7, the assumption on $u(\cdot, 0)$ and the arguments in the proof of Proposition 6.1 to derive uniform pointwise smooth estimates for u over $Y \times I_{j_0}$. If $t_{j_0+1} - t_{j_0} = 1$, we apply Lemma 6.5 and the arguments in the proof of Proposition 6.1 to achieve the same. With this, we move on to the adjacent intervals of I_{j_0} . By induction, we obtain the desired estimates for u . \square

The next lemma concerns with an energy estimate.

Lemma 6.9. *For each positive number ε , there exists a positive number ε' with the following properties. Consider a trajectory u and an interval $[T_1, T_2]$. Assume*

$$\max\{E_u(t) : t \in [T_1, T_2]\} \leq \varepsilon'.$$

Then

$$E(u, [T_1, T_2]) \leq \varepsilon.$$

Proof. Consider a sequence u_k of trajectories along with a sequence of intervals $[T_{k,1}, T_{k,2}]$ such that

$$(6.18) \quad \max\{E_{u_k}(t) : t \in [T_{k,1}, T_{k,2}]\} \rightarrow 0.$$

We claim that $E(u_k, [T_{k,1}, T_{k,2}]) \rightarrow 0$, which leads to the desired conclusion of the lemma. For simplicity, let's assume that $[T_{k,1}, T_{k,2}] \subset [T_{k+1,1}, T_{k+1,2}]$. The general case can be handled by the same argument with slight modification. The assumption (6.18) implies that in local Columb gauges, the solutions converge to Seiberg-Witten points. If these limits are all on the same Chern-Simons level, then the global energy must converge to zero in view of the equation [24, (2.18)] relating the energy to the Chern-Simons functional. Hence, if the claim is false, there must be at least two different Chern-Simons levels for the limits. By (6.18), they must be obtained at two support locations whose time distance approaches infinity. Now we consider the intermediate limits. At least one of them must lie on a Chern-Simons level different from the above two levels. Otherwise we can easily derive a contradiction to the fact that the above two levels are distinct.

But the time distance from the support location of this new limit to the support location of the previous two limits approaches infinity. Hence we have rooms in between for finding further levels. By the same arguments, these levels are pairwise different. This leads to a contradiction to the generic parameter assumption, which implies that there are only finitely many Chern-Simons levels. \square

Corollary 6.10. *There is a positive number E_0^* with the following properties. Assume that $u \in \mathcal{N}(\alpha, \beta)$ with $\alpha \neq \beta$. Then*

$$\max\{E_u(t) : t \in \mathbb{R}\} \geq E_0^*.$$

Now we proceed to analyse convergence of temporal trajectories of uniformly bounded energy. The appropriate concept of convergence is smooth piecewise exponential convergence introduced in [24, Definition 6.11]. They are measured in terms of the distance $d_{l,\mathbf{r}}$ given in [24, Definition 6.10]. Note that a pair of positive exponents $\delta = (\delta_-, \delta_+)$ are involved in this convergence concept. We shall refer to δ as the *convergence exponent*.

Theorem 6.11. *Let u_k be a sequence of smooth temporal trajectories with uniformly bounded energy. We also assume that their energies are uniformly positive. Moreover, assume that $u_k(\cdot, 0)$ are under uniform smooth control. Then there is a subsequence of u_k , still denoted u_k , along with a sequence of numbers t_k such that $\tau_{-t_k}(u_k) \equiv u_k(\cdot, \cdot + t_k)$ converge in smooth piecewise exponential fashion to a proper temporal piecewise trajectory $u = (u^1, \dots, u^m)$ with m bounded from above by a constant depending only on the energy bound. Here, the convergence exponent depends only on $K_{h,\pi,\lambda}$. If $u_k(\cdot, 0)$ are not under uniform smooth control, they can be converted into so by a sequence of gauges $g_k \in \mathcal{G}_3(Y)$.*

If $E(u_k)$ are not uniformly positive, then we have the same conclusions with $t_k = 0$ and the limit u being a time-independent smooth trajectory (in particular $m = 1$).

Proof. Consider a sequence of smooth temporal solutions u_k with uniformly bounded energy whose values at $t = 0$ are under uniform smooth control. Moreover, assume that their energies are uniformly positive. By Lemma 6.8, we have uniform pointwise smooth estimates for u_k . Choose t_k such that $E_{u_k}(t_k) \geq E_0^*$. We deal with the sequence $\tau_{-t_k}(u_k)$ instead of u_k . For convenience, we still denote it by u_k . By convergence on compact domains we obtain a limit $u_{\infty,1}$ from u_k . (We pass to subsequences whenever necessary.) We first analyse the case that the limit of energy is entirely captured by $u_{\infty,1}$, i.e.

$$E(u_{\infty,1}) = \lim E(u_k).$$

(We may assume that the limit of energy exists.) Then there is a number $T > 0$ such that for large k , the energy of u_k outside of the domain $Y \times [-T, T]$ is smaller than the critical level E_0 . By Proposition 6.1 we obtain uniform exponential estimates for u_k , which, along with the convergence on compact domains, imply uniform convergence in exponential norms with a smaller exponent. More precisely, we have

$$d_{l,\delta}(u_k, u_{\infty,1}) \rightarrow 0$$

for $\delta = (\delta_-, \delta_+)$ with $\delta_-, \delta_+ \in (0, \delta_0)$ and all l .

This is the “one piece” case of piecewise exponential convergence.

We remark in passing that the case of the energies $E(u_k)$ being not uniformly positive can be handled in a similar way.

Next assume that some energy is lost near infinity, i.e.

$$E(u_{\infty,1}) < \lim E(u_k)$$

(Compare [27] where an energy loss analysis for pseudo-holomorphic curves is presented.) Then we can find a sequence of positive numbers T_k approaching infinity such that the energy of u_k over the domain $Y \times [T_k, \infty)$ or the domain $Y \times [-\infty, -T_k)$ does not converge to zero. We handle the first, while the second is similar.

We set $t_k^1 = 0$. By the energy loss assumption and Lemma 6.9, we can find times $t_k^2 \rightarrow \infty$ such that the local energies $E_{u_k}(t_k^2)$ are uniformly positive. Now we use $-t_k^2$ as the translation amount and consider $\tilde{u}_k = \tau_{-t_k^2}(u_k)$. In other words, we translate the times t_k^2 back to zero. The limit $u_{\infty,2}$ we get from \tilde{u}_k has positive energy, and hence is nontrivial. Indeed, $E(u_{\infty,2})$ will be at least a quantum portion, namely it is no smaller than the number E_0 given in Proposition 6.1. Now we ask whether some energy might be lost between the first limit and the second. More precisely, we ask whether there are times T_k, T'_k with $T_k \rightarrow \infty, T_k < T'_k$ and $t_k^2 - T'_k \rightarrow \infty$, such that $E(u, Y \times [T_k, T'_k])$ does not converge to zero. In view of Lemma 6.9, this is equivalent to the question whether there are times t_k^3 such that $t_k^3 \rightarrow \infty, t_k^2 - t_k^3 \rightarrow \infty$ and the local energies $E_{u_k}(t_k^3)$ are uniformly positive. Assume so. Applying translations in the amount $-t_k^3$ we find a nontrivial limit. We say that the “support location” of this limit lies between those of the first and second limits. Taking into account the time order, we denote this limit by $u_{\infty,2}$, while renaming the second limit $u_{\infty,3}$.

We also ask whether there might be lost energy in support locations above that of the second limit. We consider here instead $t_k^3 \rightarrow \infty$ with $t_k^3 - t_k^2 \rightarrow \infty$.

Arguing this way, and also taking into account the $-\infty$ direction, we obtain limits $u_{\infty}^1, u_{\infty}^2, \dots$ with increasing time order for support locations. Since the energy of each limit is at least E_0 , and we obviously have

$$\sum E(u_{\infty}^j) \leq \lim E(u_k),$$

we can obtain only finitely many limits and arrive at

$$(6.19) \quad \sum_{1 \leq j \leq m} E(u_{\infty}^j) = \lim E(u_k)$$

for some m . In other words, all energy loss is accounted for. By the quantum energy property of the limits, we have an upper bound for m in terms of an energy bound.

By the energy identity (6.19), we deduce

$$(6.20) \quad \limsup_{k \rightarrow \infty} E(u_k, Y \times [t_k^j + T, t_k^{j+1} - T']) \rightarrow 0,$$

as T and T' both approach infinity.

Now we analyse the convergence of u_k to the m -tuple $(u_{\infty}^1, \dots, u_{\infty}^m)$ in more details. For simplicity, we consider the case $m = 2$. The arguments extend to the general case straightforwardly. By the energy control (6.20), we obtain for large T, T' Seiberg-Witten points u_{∞} corresponding to \tilde{u}_k along $[T, t_k^2 - T']$ as given by Lemma 6.3. By the arguments in the proof of Lemma 6.6, the Chern-Simons value $\mathbf{cs}(\infty)$ is the same for all these points. On the other hand, it is easy to see (by a simple limit argument) that

$$\mathbf{cs}(\infty) = \mathbf{cs}(\tilde{u}_k(\cdot, T_k))$$

for some $T_k \in (T, t_k^2 - T')$, as long as k is big enough. By Lemma 6.8 we obtain decay estimates

$$E_{u_k}(t) \leq C e^{-ct}$$

for $t \in [0, t_k^2/2]$, and

$$E_{\tilde{u}_k}(t) \leq Ce^{-c(t_k^2-t)}$$

for $t \in [t_k^2/2, t_k^2]$, where C depends on an energy bound. By the arguments in the proof of Proposition 6.1, we then deduce the exponential convergence of \tilde{u}_k on $Y \times (-\infty, t_k^2/2]$ to u_∞^1 with t_k^2 seen as in the positive direction, and on $Y \times [t_k^2/2, +\infty)$ to u_∞^2 , with $t_k^2/2$ seen as in the negative direction. We also derive for $t, t' > 0$

$$\|\tilde{u}_k(\cdot, t) - \tilde{u}_k(\cdot, t_k^2 - t')\|_{L^\infty} \leq C(e^{-ct} + e^{-ct'}).$$

Taking limit, we deduce that

$$\|u_\infty^1(\cdot, t) - u_\infty^2(\cdot, -t')\|_{L^\infty} \leq C(e^{-ct} + e^{-ct'}).$$

This implies that $\lim_{t \rightarrow +\infty} u_\infty^1 = \lim_{t \rightarrow -\infty} u_\infty^2$. Consequently, (u_∞^1, u_∞^2) is a piecewise trajectory, cf. [24, Definitions 6.5]. It is proper because both u_∞^1 and u_∞^2 have nonzero energy. By the obtained estimates one readily deduces that u_k converges to (u_∞^1, u_∞^2) in smooth piecewise exponential fashion. \square

Proof of [24, Theorem 6.15]. The compactness part is an immediate consequence of Theorem 6.11. To show the Hausdorff property of $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$, consider $\omega_1 = (\omega_1^1, \dots, \omega_1^k) \in \underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k, \omega_2 = (\omega_2^1, \dots, \omega_2^j) \in \underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_j, \omega_1 \neq \omega_2$. Choose piecewise trajectories u_0, v_0 to represent ω_1 and ω_2 respectively. For $r > 0, \varepsilon > 0$ we set $\mathbf{r}_k = (r, \dots, r) \in \mathbb{R}_+^k$ and define

$$U_r(\omega_1) = \{[u]_0^T \in \mathcal{M}_T^0(S_\alpha, S_\beta) : \text{There is a } u \in [u]_0^T \text{ such that} \\ d_{2, \mathbf{r}_k}(u, u_0 \# \mathbf{r}_k) < \varepsilon\},$$

where $\#$ is the suspension (pre-gluing) map introduced in [24, Section 6]. Similarly, we have $U_r(\omega_2)$. For r large enough and ε small enough, the neighborhoods $\bar{U}_r(\omega_1)$ and $\bar{U}_r(\omega_2)$ are disjoint. \square

The proof of [24, Proposition 6.18] is similar, we leave the details to the reader.

7. STRUCTURES NEAR INFINITY

The main purpose of this section is to establish the smooth structure of the moduli spaces $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ (the temporal model), which is equivalent to the smooth structure of the moduli spaces $\underline{\mathbf{M}}^0(p, q; \mathcal{SW}_0)$ (the fixed-end model) with $p \in \alpha, q \in \beta$ (see [24, Section 6]). We show namely that they are smooth manifolds with corners. In other words, they are modelled on $\bar{\mathbb{R}}_+^d$, where d is the relevant dimension. (Recall $\mathbb{R}^+ = (0, \infty)$.) There is a natural structure of stratification for these moduli spaces, which is provided by its subspaces of k -trajectories. More precisely, we have

$$\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta) = \cup_k \underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k, \\ \underline{\mathbf{M}}^0(p, q; \mathcal{SW}_0) = \cup_k \underline{\mathcal{M}}^0(p, q; \mathcal{SW}_0)_k,$$

cf. [24, Section 6]. To establish the said smooth structure, it remains to construct coordinate charts along these strata. We present a detailed treatment of the temporal model. We also sketch a treatment of the fixed-end model (independent of the treatment of the temporal model). Of course, we only need either one for the constructions of our Seiberg-Witten Floer theories. However, it is useful for conceptual understanding to clarify the both models.

In the following, we assume that the relevant perturbation parameters are generi-

Proposition 7.1. *Consider (distinct) $\alpha, \beta \in \mathcal{R}$. There are neighborhoods U_k, \hat{U}_k of $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k$ in $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k \times [0, \infty)^{k-1}$ and $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ respectively, and a homeomorphism $\mathbf{F}_k : U_k \rightarrow \hat{U}_k$ such that the restriction of \mathbf{F}_k to $U_k^0 = U_k \cap (\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k \times (0, \infty)^{k-1})$ is a diffeomorphism. Moreover, the following hold:*

(1) *The restriction of \mathbf{F}_k to $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k$, which is identified with $\mathcal{M}_T^0(S_\alpha, S_\beta)_k \times \{(0, \dots, 0)\}$, is the identity map.*

(2) *For each compact set K in $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k$, there is a positive number r_0 such that $K \times [0, r_0]^{k-1} \subset U_k$.*

(3) *For $1 \leq j \leq k-1$, the restriction of \mathbf{F}_k to the j -th open boundary stratum of U_k is a diffeomorphism onto the j -th open boundary stratum $\hat{U}_k \cap \underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_j$ of $\hat{U}_k \cap \underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$. Here e.g. the first boundary stratum of $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k \times [0, \infty)^{k-1}$ is $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k \times (\{0\} \times (0, \infty)^{k-2} \cup (0, \infty) \times \{0\} \times (0, \infty)^{k-2} \dots \cup (0, \infty)^{k-2} \times \{0\})$.*

(4) *For different k, j , the transitions between \mathbf{F}_k and \mathbf{F}_j are smooth.*

The maps \mathbf{F}_k define the structure of smooth manifolds with corners for the temporal model $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ as stated in [24, Theorem 6.19].

Obviously, the crucial result [24, Theorem 6.19] on the smoothness of $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ follows from this proposition.

The maps \mathbf{F}_k will be constructed by a gluing process. First, we need to construct suitable presentation models for our moduli spaces.

Consider a temporal Seiberg-Witten trajectory $u \in \mathcal{N}_T(S_\alpha, S_\beta)$ (see [24, Section 6]) for distinct α and β . We define $\rho_+(u)$ and $\rho_-(u)$ by the following equations

$$E(u, Y \times [\rho_+(u), \infty)) = E_0/2, E(u, Y \times (-\infty, \rho_-(u))) = E_0/2,$$

where E_0 is given in Proposition 6.1. Let $\langle u \rangle$ denote the set of trajectories which are gotten from u by a time translation. Let $u_+^*(u)$ denote the element in $\langle u \rangle$ whose ρ_+ value equals zero, and $u_-^*(u)$ denote the one whose ρ_- value equals zero.

Lemma 7.2. *The elements $u_+^*(u)$ and $u_-^*(u)$ are uniquely determined. Moreover, they each give rise to a transversal slice for the time translation action on $\mathcal{N}_T(S_\alpha, S_\beta)$.*

Proof. First, we show that e.g. $u_-^*(u)$ is well-defined. It is easy to see that there is some $\tilde{u} = \tau_t(u)$ in $\langle u \rangle$ such that $E(\tilde{u}, Y \times [0, \infty)) = E_0/2$. If there is another, then it follows that the energy of u over $Y \times I$ for a nontrivial interval I is zero. By unique continuation (see below for more details), $\partial u / \partial t$ must be identically zero. This is impossible, because $\alpha \neq \beta$.

Next we consider the function $E(u, [0, \infty))$ on $\mathcal{N}_T(S_\alpha, S_\beta)$. We have

$$\frac{d}{dt} E(\tau_t(u), [0, \infty))|_{t=0} = \frac{d}{dt} E(u, [t, \infty))|_{t=0} = \int_{Y \times \{0\}} \left| \frac{\partial u}{\partial t} \right|^2.$$

If this is zero, then $\partial u / \partial t$ is identically zero at $t = 0$. Differentiating the equation $\partial u / \partial t = \mathbf{sw}(u)$ we derive that all time derivatives of u at $t = 0$ vanish. By unique continuation, this implies that $\partial u / \partial t \equiv 0$, which is impossible. More precisely, we consider the quantity $v = \mathbf{sw}(u)$. First, we know that $v = 0$ at time $t = 0$. But $\partial v / \partial t = d\mathbf{sw}(v) = 0$ at $t = 0$. Arguing in this fashion we deduce that all space and time derivatives of v at $t = 0$ vanish. On the other hand, v satisfies the elliptic system (6.15) or rather (6.16), which implies a second order system with scalar symbol. Hence the unique continuation principle holds. It follows that $v = 0$.

We conclude that the derivative $dE(\tau_t(u), [0, \infty))/dt \neq 0$. The desired transversality follows. \square

Let $\mathcal{N}_T^s(S_\alpha, S_\beta)$ denote the transversal slice given by u_-^* , i.e. $\mathcal{N}_T^s(S_\alpha, S_\beta)$ is the submanifold of $\mathcal{N}_T(S_\alpha, S_\beta)$ defined by the equation $\rho_- = 0$. We have corresponding slices $\mathcal{N}_T^s(S_\alpha, S_\beta)_k, \mathbf{N}_T^s(S_\alpha, S_\beta)$ for the corresponding spaces of piecewise trajectories. By the gauge invariance of energy, all these slices are preserved under actions by gauges in $\mathcal{G}_3(Y)$. (Because of the temporal condition, only such gauges are admitted.)

Next we construct a global slice for the action of based gauges. Consider the global slice \mathcal{S}_1 for the action of $\mathcal{G}_3^0(Y \times [-1, 1])$ on $\mathcal{A}_2(Y \times [-1, 1]) \times \Gamma^+(Y \times [-1, 1])$ introduced in [24, Lemma 8.2]. Using the temporal transformation g_T introduced in [24, Lemma 6.1] we convert \mathcal{S}_1 into a global slice $\tilde{\mathcal{S}}_1$ consisting of temporal elements. We set

$$(7.1) \quad \mathcal{S}_T(\alpha, \beta) = \{u \in \mathcal{N}_T^s(S_\alpha, S_\beta) : u|_{Y \times [-1, 1]} \in \tilde{\mathcal{S}}_1\}.$$

The following lemma is readily proved.

Lemma 7.3. *The space $\mathcal{S}_T(\alpha, \beta)$ is a global transversal slice of the action of $\mathcal{G}_3^0(Y)$ on $\mathcal{N}_T^s(S_\alpha, S_\beta)$. Consequently, it is diffeomorphic to the moduli space $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)$ and hence can be used as its presentation model. By the construction and the arguments at the beginning of Section 6, we have smooth control over $u(\cdot, 0)$ for $u \in \mathcal{S}_T(\alpha, \beta)$.*

Next we construct presentation models for the moduli spaces of consistent multiple temporal trajectory classes $\mathcal{M}_T^0(S_{\alpha_0}, \dots, S_{\alpha_k})$. First, we have the fibered product

$$\begin{aligned} \mathcal{S}_T(\alpha_0, \alpha_1) \times_{S_{\alpha_1}} \mathcal{S}_T(\alpha_1, \alpha_2) \times_{S_{\alpha_2}} \dots \mathcal{S}_T(\alpha_{k-1}, \alpha_k) = \\ \{(u_1, \dots, u_k) \in \mathcal{S}_T(\alpha_0, \alpha_1) \times \dots \mathcal{S}_T(\alpha_{k-1}, \alpha_k) : \\ \pi_+(u_i) = \pi_-(u_{i+1}), i = 1, \dots, k-1\} \end{aligned}$$

as a presentation model. However, the elements here may not be piecewise trajectories, i.e. the endpoints of u_i as above may not match each other. We modify this model as follows. Let End_\pm denote the endpoint (i.e. limit) at $\pm\infty$. For $u = (u_1, \dots, u_k)$ in this model, we set $\tilde{u} = (u_1, u_2 + End_+(u_1) - End_-(u_2), \dots, u_k + \sum_{1 \leq i \leq k-1} (End_+(u_i) - End_-(u_{i+1})))$. Obviously, \tilde{u} is a temporal piecewise trajectory. By the consistent condition, the adjustments involved in each portion are in terms of actions of elements in $\mathcal{G}_3^0(Y)$. Hence we obtain our desired better representation model for $\underline{\mathcal{M}}_T^0(S_{\alpha_0}, \dots, S_{\alpha_k})$:

$$\mathcal{S}_T(\alpha_0, \dots, \alpha_k) = \{\tilde{u} : u \in \mathcal{S}_T(\alpha_0, \alpha_1) \times_{S_{\alpha_1}} \mathcal{S}_T(\alpha_1, \alpha_2) \times_{S_{\alpha_2}} \dots \mathcal{S}_T(\alpha_{k-1}, \alpha_k)\}.$$

By taking unions we then obtain representation model $\mathcal{S}_T(\alpha, \beta)_k$ for $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)_k$, and representation model $\mathbf{S}_T(\alpha, \beta)$ for $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$. We set

$$\tilde{\mathcal{S}}_T(\alpha, \beta) = \cup_k (\mathcal{S}_T(\alpha, \beta)_k \times \mathbb{R}_+^{k-1}).$$

The following lemma follows from straightforward computations. Here, we use again the suspension or pre-gluing operator $\#$ introduced in [24, Section 6]. The piecewise exponential weight $w_{u\#r} = w_r$ with $r = (r_1, \dots, r_{k-1}) \in \mathbb{R}_+^{k-1}$ and the corresponding weighted norm $\|\cdot\|_{l, w_{u\#r}}$ were also introduced in [24, Section 6]. These quantities involve a pair of exponents $\delta = (\delta_-, \delta_+)$. Throughout the sequel we fix a pair δ with $\delta_-, \delta_+ \in (0, \delta_-)$, where δ_- is given in Proposition 6.1.

Lemma 7.4. *Let K be a compact set in $\mathcal{S}_T(\alpha, \beta)_k$. Then there are constants $C = C(K)$ and $r_0 = r_0(K)$ such that*

$$(7.2) \quad \|\mathbf{SW}(u\sharp\mathbf{r})\|_{1, w_{u\sharp\mathbf{r}}} \leq Ce^{-(\delta_0 - \delta)r_{\min}}$$

for $u \in K, r_i \geq r_0$, where $r_{\min} = \min \{r_1, \dots, r_{k-1}\}$.

Proposition 7.5. *Let K be a compact set in $\mathcal{S}_T(\alpha, \beta)_k$. There are positive numbers $\bar{r} = \bar{r}(K)$ and $\sigma = \sigma(K)$ with the following properties. For $u \in K, \mathbf{r} = (r_1, \dots, r_{k-1}) \in \mathbb{R}_+^{k-1}$ and $\bar{u} = (u, \mathbf{r})$, consider the linearization of \mathbf{SW} at $\sharp\bar{u} \equiv u\sharp\mathbf{r}$, $d\mathbf{SW}_{\sharp\bar{u}} : \Omega_{2,\delta}^1 \oplus \Gamma_{2,\delta}^+ \rightarrow \Omega_{1,\delta}^+ \oplus \Gamma_{1,\delta}^-$. If*

$$(7.3) \quad r_i \geq \bar{r}, i = 1, \dots, k-1,$$

then $d\mathbf{SW}_{\sharp\bar{u}}$ has a right inverse $Q = Q_{\bar{u}}$ with

$$(7.4) \quad \|Q\|_{w_{\bar{u}}} \leq C = C(K),$$

where the subscript $w_{\bar{u}}$ means that the norm is measured in terms of the $w_{\bar{u}}$ -weighted Sobolev norms instead of the δ -weighted norms, cf [24, Section 6]. Furthermore, the equation $\mathbf{SW}(\sharp\bar{u} + Q\zeta) = 0$ with $\zeta \in \Omega_{1,\delta}^+ \oplus \Gamma_{1,\delta}^-, \|\zeta\|_{1, w_{\bar{u}}} \leq \sigma$ has a unique solution $\zeta(\bar{u})$.

For $\bar{u} = (u, \mathbf{r})$ with $r_1, \dots, r_{k-1} \geq \bar{r}$ we define

$$\mathcal{G}l_K(\bar{u}) = \sharp\bar{u} + Q\zeta(\bar{u})$$

and

$$\mathcal{G}l_K^T(\bar{u}) = T_G(\mathcal{G}l_K(\bar{u})),$$

where T_G is the temporal transformation, see [24, Section 6].

For \bar{u} with $r_1, \dots, r_{k-1} \in (0, \bar{r}^{-1}]$ we define

$$\tilde{\mathcal{G}}l_K^T(\bar{u}) = \mathcal{G}l_K^T(\text{Inv}(\bar{u})),$$

where

$$\text{Inv}(\bar{u}) = (u, (r_1^{-1}, \dots, r_{k-1}^{-1})).$$

These gluing maps for different K coincide with each other on the intersection of their definition domains, hence we shall replace the subscript K by k . Thus we have $\mathcal{G}l_k, \mathcal{G}l_k^T$ and $\tilde{\mathcal{G}}l_k^T$. Taking an exhaustion of compact domains $K_0 = \emptyset \subset K_1 \subset K_2 \cdots$ for $\mathcal{S}_T(\alpha, \beta)^k$, we then obtain a gluing map $\tilde{\mathcal{G}}l_k^T : U_T(\alpha, \beta)_k \rightarrow \mathcal{N}_T(\alpha, \beta)$, where the domain of gluing $U_T(p, q)_k \subset \mathcal{S}_T(\alpha, \beta)_k \times \mathbb{R}_+^{k-1}$ is defined by the condition

$$r_i \in (0, \bar{r}^{-1}(K_j)]$$

for (u, \mathbf{r}) with $u \in K_j - K_{j-1}$. This is a smooth map.

Proof. We present the case $k = 2$. The other cases can be treated in a similar way. Consider $u = (u^1, u^2) \in \mathcal{S}_T(\alpha, \beta)_2$ and $r \in \mathbb{R}_+$. For the right inverse Q given in [24, Lemma 4.16] we note the following identity

$$(7.5) \quad \|\mathcal{Q}\|_{w_{\bar{u}}} \leq C = C(K)$$

We set $Q_1 = Q_{u^1}, Q_2 = Q_{\tau_r(u^2)}$ and define $\tilde{Q} : \Omega_{1,\delta}^+ \oplus \Gamma_{1,\delta}^- \rightarrow \Omega_{2,\delta}^+ \oplus \Gamma_{2,\delta}^-$ by

$$\tilde{Q}(\zeta) = \tau_r(\eta_1)Q_1(\tau_r(\eta_1)\zeta) + \tau_r(\eta_2)Q_2(\tau_r(\eta_2)\zeta).$$

Then we have for $\bar{u} = u\sharp R$ and $w = w_{\bar{u}}$

$$\|d\mathbf{SW}_{\bar{u}} \circ \tilde{Q} - Id\|_w \leq Ce^{-(\delta_0 - \delta)r}.$$

Hence we obtain for large r a right inverse $Q = \tilde{Q} \circ (d\mathbf{SW}_{\bar{u}} \circ \tilde{Q})^{-1}$ of $d\mathbf{SW}_{\bar{u}}$. By [24, Lemma 4.16], we have the desired norm estimate for Q .

By the implicit function theorem, $\mathbf{SW}(\bar{u} + Q)$ is a diffeomorphism from a neighborhood U of $0 \in \Omega_{1,\delta}^+ \oplus \Gamma_{1,\delta}^-$ onto a neighborhood \tilde{U} of $\mathbf{SW}(\bar{u}) \in \Omega_{1,\delta}^+ \oplus \Gamma_{1,\delta}^-$, where the size of U, \tilde{U} depends only on K , and is independent of r . By Lemma 7.4, 0 must be contained in \tilde{U} if r is large enough. \square

Descending to quotient, the gluing maps $\mathcal{G}l_k^T$ and $\tilde{\mathcal{G}}l_k^T$ induce gluing maps $[\mathcal{G}l_k^T]$ and $[\tilde{\mathcal{G}}l_k^T]$ into $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)$. (For simplicity, we omit the indication of the time translation quotient in the notations $[\mathcal{G}l_k^T]$ and $[\tilde{\mathcal{G}}l_k^T]$.) On the other hand, by a simple limit process we extend $\tilde{\mathcal{G}}l_k^T$ to $\bar{U}_T(\alpha, \beta)_k \subset \mathcal{S}_T(\alpha, \beta)_k \times [0, \infty)^{k-1}$. Correspondingly, we extend $[\tilde{\mathcal{G}}l_k^T]$ to $\bar{U}_T(\alpha, \beta)_k$. This extended map is a continuous map into $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)$. It is smooth in the interior. Moreover, its restriction to each open boundary stratum of $\bar{U}_T(\alpha, \beta)_k$ is a smooth map into the corresponding stratum of $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)$.

The behavior of $\tilde{\mathcal{G}}l_k^T$ along the boundary strata of $\bar{U}_T(\alpha, \beta)_k$ is easy to analyse. Indeed, the relations given in the following lemma are easy consequences of the construction of the gluing maps.

Lemma 7.6. *We have*

$$(7.6) \quad \tilde{\mathcal{G}}l_k^T(u, (0, \dots, 0)) = u,$$

$$(7.7) \quad \tilde{\mathcal{G}}l_k^T((u_1, \dots, u_k), (0, r_2, \dots, r_{k-1})) = (g(u_2, \dots, u_k, r_2, \dots, r_{k-1}))^* u_1,$$

$$\tilde{\mathcal{G}}l_{k-1}^T((u_2, \dots, u_k), (r_2, \dots, r_{k-1})),$$

etc., where $g(u_2, \dots, u_k, r_2, \dots, r_{k-1}) \in \mathcal{G}(Y)$ is determined by the temporal transformation from $\mathcal{G}l_{k-1}$ to $\mathcal{G}l_{k-1}^T$.

Before proceeding, we introduce some notations and terminologies. First, the space $L_{2,\delta}(\mathcal{A}_2(Y) \times \Gamma_2(Y), \mathcal{A}_2(Y) \times \Gamma_2(Y))$ (cf. [24, Section 6]) has a natural double affine structure given by

$$L_{2,\delta}(\mathcal{A}_2(Y) \times \Gamma_2(Y), \mathcal{A}_2(Y) \times \Gamma_2(Y)) \cong H,$$

where

$$(7.8) \quad H = (\oplus_{i=1}^k (\Omega^1(Y) \oplus \Gamma_1(Y))) \oplus (\Omega^1(Y) \oplus \Gamma_1^+(Y))$$

and the correspondence is given by

$$(7.9) \quad u \rightarrow \Phi(u) = (End_-(u) - u_0, End_+(u) - u_0, u - (u_0 + \chi(End_-(u) - u_0) + (1 - \chi)(End_+(u) - u_0)))$$

with χ being the cut-off function introduced in [24, Section 4]. The distance $d_{2,\delta}$ given in [24, Definition 4.6] can be thought of as induced from the natural norm on H via this correspondence.

Next consider

$$L_{2,\delta}(\mathcal{A}_2(Y) \times \Gamma_2(Y), \mathcal{A}_2(Y) \times \Gamma_2(Y))_k \equiv \{(u_1, \dots, u_k) \in L_{2,\delta}(\mathcal{A}_2(Y) \times \Gamma_2(Y), \mathcal{A}_2(Y) \times \Gamma_2(Y))^k : End_+(u_j) = End_-(u_{j+1})\}.$$

The distance $d_{2,\delta}$ naturally extends to this space, which we still denote by $d_{2,\delta}$. On the other hand, there is a correspondence Φ_k similar to Φ with the model H replaced by

$$(7.10) \quad H_k = (\oplus_{1 \leq j \leq k+1} (\Omega_2^1(Y) \oplus \Gamma_2(Y))) \oplus (\oplus_{1 \leq j \leq k} (\Omega_{2,\delta}^1(X) \oplus \Gamma_{2,\delta}^+(X))).$$

The norm on H_k will be denoted by $\|\cdot\|_{H_k}$.

The distance $d_{2,\delta}$ descends to $\mathcal{S}_T(\alpha, \beta)_k$, which is a submanifold of $L_{2,\delta}(\mathcal{A}_2(Y) \times \Gamma_2(Y), \mathcal{A}_2(Y) \times \Gamma_2(Y))_k$, and induces its natural topology. We also obtain the tangent spaces of $\mathcal{S}_T(\alpha, \beta)_k$ as suitable subspaces of H_k .

We have the product distance on $L_{2,\delta}(\mathcal{A}_2(Y) \times \Gamma_2(Y), \mathcal{A}_2(Y) \times \Gamma_2(Y))_k \times \mathbb{R}_+^{k-1}$, denoted again by $d_{2,\delta}$, and product norm on $H_k \oplus \mathbb{R}^{k-1}$, denoted by $\|\cdot\|_{H_k \oplus \mathbb{R}^{k-1}}$. They descend to $\mathcal{S}_T(\alpha, \beta)_k \times \mathbb{R}_+^{k-1}$ and its tangent spaces respectively.

Next we introduce another norm which corresponds to the distance $d_{2,\mathbf{r}}$ introduced in [24, Definition 6.12] for measuring piecewise exponential convergence.

Definition 7.7. For $\mathbf{r} \in \mathbb{R}_+^{k-1}$ and $(w_1, w_2, v) \in T_u \mathcal{S}_T(\alpha, \beta)$ we set

$$(7.11) \quad \|(w_1, w_2, v)\|_{\mathbf{r}}^2 = \sum_{1 \leq j \leq 2} \|w_i\|_2^2 + \sum_{1 \leq j \leq k-1} \|v(\cdot, r_i)\|_2^2 + \|v - Int_{\mathbf{r}}^*(v)\|_{2, w_{\mathbf{r}}}^2,$$

where

$$Int_{\mathbf{r}}^*(v) = \tau_{r_1}(\eta)v(\cdot, r_1) + \dots + \tau_{2r_1 + \dots + 2r_{k-2} + r_{k-1}}(\eta)v(\cdot, 2r_1 + \dots + 2r_{k-2} + r_{k-1})$$

and η is the cut-off function introduced in [24, Section 6].

Lemma 7.8. $[\tilde{\mathcal{G}}l_k^T]$ is a local diffeomorphism along $U_T(\alpha, \beta)_k$. Equivalently, $[\mathcal{G}l_k^T]$ is a local diffeomorphism along the domain corresponding to $U_T(\alpha, \beta)_k$ (the correspondence is in terms of the map Inv). Indeed, for each compact set K in $\mathcal{S}_T(\alpha, \beta)_k$, there are positive numbers r_K, R_K and ρ_K with the following property. Let $u \in K$ and $\mathbf{r} = (r_1, \dots, r_{k-1}) \in \mathbb{R}_+^{k-1}$ with $r_i \in (r_K, \infty), i = 1, \dots, k-1$. Then there is a neighborhood $U(u, \mathbf{r})$ of (u, \mathbf{r}) such that the restriction of $[\mathcal{G}l_k^T]$ to it is a diffeomorphism onto the distance ball $B_{\rho_K}([\mathcal{G}l_k^T](u, \mathbf{r}))$, where the distance is the $(2, \mathbf{r})$ -distance $d_{2,\mathbf{r}}$. Moreover, $U(u, \mathbf{r})$ contains $B_{r_K}(u) \times B_{R_K}(\mathbf{r})$, where the distance is $d_{2,\delta}$.

Proof. Since $\mathcal{S}_T(\alpha, \beta)$ is a presentation model for $\underline{\mathcal{M}}_T^0(S_\alpha, S_\beta)$, we obtain from $[\mathcal{G}l_k^T]$ an induced solving map $\mathcal{G}l_k^0 : U_T(\alpha, \beta)_k \rightarrow \mathcal{S}_T(\alpha, \beta)_k$. We show that $\mathcal{G}l_k^0$ is a local

diffeomorphism with size control corresponding to the size control stated in the lemma.

Fix a compact domain K in $\mathcal{S}_T(\alpha, \beta)_k$, e.g. $K = K_j$ for some j . Consider $u \in K$ and $\mathbf{r} \in \mathbb{R}_+^{k-1}$ with $r_i \in [2\bar{r}_K, \infty)$. We write $u^* = \mathcal{G}l_k^0(u, \mathbf{r})$. Solving the equation $\mathbf{SW}(\Phi_k^{-1}(u + v + v^\perp)) = 0$ (this means $\mathbf{SW} = 0$ for each portion in $u + v + v^\perp$) for $v \in T_u \mathcal{S}_T(\alpha, \beta)_k$, $v^\perp \in T_u \mathcal{S}_T(\alpha, \beta)_k$ by using the implicit function theorem and elliptic estimates, we obtain a coordinate map $\Phi_u : B_{r_K}(0) \rightarrow U(u)$ with smooth control, such that $U(u)$ contains the ball $B_{\rho_K}(u)$. Similarly, we obtain a coordinate map $\Psi_{u^*} : B_{r_K^*}^*(0) \rightarrow U(u^*)$ with smooth control, where $U(u^*)$ contains the ball $B_{\rho_K^*}^*(u^*)$. Here, the superscript \mathbf{r} means that the ball is a distance ball with respect to the norm $\|\cdot\|_{\mathbf{r}}$ or the distance $d_{2, \mathbf{r}}$.

By the construction of $\mathcal{G}l_k^0$ and elliptic estimates, we can choose the numbers r_K, r_K^*, ρ_K and ρ_K^* suitably, such that the composition map

$$\mathcal{G}l^u \equiv \Psi_{u^*}^{-1} \cdot \mathcal{G}l_k^0 \cdot (\Phi_u \times Id)$$

goes from $B_{r_K} \times B_{\bar{r}_K/2}(\mathbf{r})$ into $B_{r_K^*}^*(0)$.

Now we proceed to compute the differential of $\mathcal{G}l^u$. For this purpose, we introduce another gluing map. For $v \in B_{r_K}(0)$ and $\mathbf{r}' \in B_{r_K}(\mathbf{r})$, we consider the equation

$$\mathbf{SW}(\#(\Phi_k^{-1}(v), \mathbf{r}') + Qv') = 0.$$

By the proof of Lemma 7.4 we obtain solutions

$$v' = \zeta^*(v, \mathbf{r}')$$

for v, \mathbf{r}' in balls of size depending only on K . We set

$$(7.12) \quad \sigma(v, \mathbf{r}') = \#(\Phi_k^{-1}(v), \mathbf{r}') + Q\zeta^*(v, \mathbf{r}').$$

Converting σ into the slice $\mathcal{S}_T(\alpha, \beta)$ by a transformation P involving the time translation and gauge transformations, we obtain a new map which we denote by $\mathcal{G}l_0^u$. By tangency, the differential $d\mathcal{G}l_0^u|_u$ equals the differential $d\mathcal{G}l^u|_u$. Since the transformation P can easily be controlled, it suffices for the purpose of computing $d\mathcal{G}l^u|_u$ to compute $d\sigma|_u$.

Consider e.g. the case $k = 2$. The general case is similar. Consider the above $u = (u_1, u_2)$ and $\mathbf{r} = r$. The contribution of $Q\zeta^*$ to Σ is small and can be absorbed into that of the first term. Thus, we have

$$(7.13) \quad d\sigma|_{(u, \mathbf{r})}(0, (v_1, v_2), 0) \cong (v_1, 0)\sharp r + (0, v_2)\sharp r.$$

Using the exponential decay properties of the tangent vectors $v = (v_1, v_2)$ and the finite dimensionality of the tangent space we deduce

$$(7.14) \quad \|d\sigma|_{(u, \mathbf{r})}(0, (v_1, v_2), 0)\|_{\mathbf{r}} \geq c(K)\|(v_1, v_2)\|_{2, \delta}$$

for a positive constant $c(K)$. Here the finite dimensionality is used in the following way: we consider an orthonormal base of tangent vectors and derive the estimate (7.14) for them first. Then (7.14) follows for general tangent vectors $(0, v, v, 0)$.

On the other hand, we have

$$(7.15) \quad d\sigma|_{(u,\mathbf{r})}(0,0,1) \cong \tau_r\left(\frac{\partial u_2}{\partial t}\right).$$

Using the arguments in the proof of Lemma 7.2 and compactness we derive a positive lower bound for $\int_{t=0} |\partial u_2 / \partial t|^2$. Consequently, we deduce a positive lower bound on $\|d\Sigma|_{(u,\mathbf{r})}(0,0,1)\|_{2,\mathbf{r}}$. Combining this with (7.11) and applying the transversality of the model $\mathcal{S}_T(\alpha, \beta)$ with respect to the time translation action, we then deduce

$$(7.16) \quad \|d\Sigma|_{(u,\mathbf{r})}(0, (v_1, v_2), r')\|_{\mathbf{r}} \geq c(K)(\|(v_1, v_2)\|_{2,\delta}^2 + (r')^2)^{1/2}.$$

Finally, we have

$$(7.17) \quad d\sigma|_{(u,\mathbf{r})}(w_1, w_2, w_3, 0, 0, 0) \cong \chi w_1 + \tau_r(\eta)w_2 + (1 - \chi)w_3.$$

Combining the above estimates and arguments we arrive at

$$(7.18) \quad \|d\Sigma|_{(u,\mathbf{r})}\| \geq c(K) > 0,$$

where the operator norm is defined with respect to the norms $\|\cdot\|_{H_k \oplus \mathbb{R}_+^{k-1}}$ and $\|\cdot\|_{\mathbf{r}}$. Note that it is easy to handle the cross interactions between (w_1, w_2, w_3) and (v_1, v_2, r') here. The estimate (7.18) immediately leads to

$$(7.19) \quad \|d\mathcal{G}l^u|_{(u,\mathbf{r})}\| \geq c(K) > 0.$$

We can compute the differential $d\mathcal{G}l^u$ at other points in a similar way and derive an upper bound on its norm. It involves only minor modifications of the above computations.

By the implicit function theorem, $\mathcal{G}l^u$ is a diffeomorphism from a neighborhood of $(0, \mathbf{r}, 0)$ onto a neighborhood of 0 with the desired size control. This implies that $\mathcal{G}l_k^0$ is a diffeomorphism from a neighborhood of (u, \mathbf{r}) onto a neighborhood of u^* with the desired size control. The desired properties of $[\mathcal{G}l_k^T]$ follow. \square

Lemma 7.9. *The restriction of $[\tilde{\mathcal{G}}l_k^T]$ to each open stratum of $\bar{U}_T(\alpha, \beta)_k$ is a local diffeomorphism into the corresponding open stratum of $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ with size control similar to that in Lemma 7.8.*

The poof of this lemma is similar to that of Lemma 7.8, we omit the details.

Lemma 7.10. *We can choose the domain $U_T(\alpha, \beta)_k$ suitably such that $[\tilde{\mathcal{G}}l_k^T]$ is a homeomorphism from $\bar{U}_T(\alpha, \beta)_k$ onto a neighborhood of $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)_k$ in the moduli space $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$.*

Proof. First we show that the image of $\bar{U}_T(\alpha, \beta)_k$ is a neighborhood of $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)_k$ in $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$. Assume the contrary. Then there would be a sequence ω_j in $\underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$ converging to some $\omega \in \underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)_k$ such that none of ω_j is in the image of $[\tilde{\mathcal{G}}l_k^T]$. We consider the case $\omega_j \in \underline{\mathbf{M}}_T^0(S_\alpha, S_\beta)$, while the other cases are similar. We have representatives $u_j \in \omega_j, u \in \omega$ such that $u_j \in \mathcal{S}_T(\alpha, \beta), u \in \mathcal{S}_T(\alpha, \beta)$. By Lemma 7.2, $\tau_r(\alpha_j)$ converge to α piecewise exponentially, where

t_k is a suitable sequence of numbers. Hence there is a sequence $\mathbf{r}_j = (r_{k,1}, \dots, r_{j,k-1})$ such that

$$d_{2,\mathbf{r}_j}(\tau_{-t_k}(u_j), u \# \mathbf{r}_j) \rightarrow 0.$$

It follows then that

$$d_{2,\mathbf{r}_j}(\omega_j, [\mathcal{G}l_k^T](u, \mathbf{r}_j)) \rightarrow 0.$$

Applying Lemma 7.8 we deduce that for large j , ω_j is in the image of $[\mathcal{G}l_k^T]$, and hence in the image of $[\tilde{\mathcal{G}}l_k]$. This is a contradiction.

Next we show that $[\mathcal{G}l_k^T]$ is injective. First note that the images of different strata of $\bar{U}_T(\alpha, \beta)_k$ under $[\tilde{\mathcal{G}}l_k^T]$ are disjoint. Hence we can consider each stratum individually. Consider e.g. the top stratum, which is contained in $\mathcal{S}_T(\alpha, \beta)_k \times \mathbb{R}_+^{k-1}$. Lower strata can be handled in a similar way, where we apply Lemma 7.9 instead of Lemma 7.8. Assume $[\tilde{\mathcal{G}}l_k^T](u, \mathbf{r}) = [\tilde{\mathcal{G}}l_k^T](\tilde{u}, \tilde{\mathbf{r}})$ for $u, \tilde{u} \in \mathcal{S}_T(\alpha, \beta)_k$, $\mathbf{r}, \tilde{\mathbf{r}} \in \mathbb{R}_+^{k-1}$. Consider the path $\theta(t) = (u(t), \mathbf{r}(t)) = (u, t\mathbf{r})$, $t \in [0, 1]$. By Lemma 7.8 we can find a path $\tilde{\theta}(t) = (\tilde{u}(t), \tilde{\mathbf{r}}(t))$, $t \in (0, 1]$ such that $\tilde{\theta}(1) = (\tilde{u}, \tilde{\mathbf{r}})$ and $[\tilde{\mathcal{G}}l_k^T](\tilde{\theta}(t)) = [\tilde{\mathcal{G}}l_k^T](\theta(t))$. By the identity (7.6) in Lemma 7.6, we deduce that the path $\tilde{\theta}$ extends continuously to $\tilde{\theta}(0) = (u, (0, \dots, 0))$.

We claim that for t small enough, $\tilde{\theta}(t) = \theta(t)$. For simplicity, we consider the case $k = 2$. The general case is similar. In this case we have $u = (u_1, u_2)$, $\mathbf{r} = r$, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$, $\tilde{\mathbf{r}} = \tilde{r}$ etc.

There is a unique number $R_{u_1}^+$ such that $E(u_1, Y \times [R_{u_1}^+, \infty)) = E_0/8$ and $E(u_1, Y \times (-\infty, R_{u_1}^+]) \geq 7E_0/8$. Similarly, there is a unique number $R_{u_2}^-$ such that $E(u_2, Y \times (-\infty, R_{u_2}^-]) = E_0/8$ and $E(u_2, [R_{u_2}^-, \infty)) \geq 7E_0/8$. It follows that $u \# (tr)^{-1}$ has an “energy valley” of length $2(tr)^{-1} - R_{u_1}^+ - R_{u_2}^- + o(1)$ with $o(1) \rightarrow 0$ as $r \rightarrow 0$. Here an “energy valley” of $u \# (tr)^{-1}$ means an interval $I = [R_1, R_2]$ such that

$$E(u \# (tr)^{-1}, Y \times I) = E_0/4,$$

$$E(u \# (tr)^{-1}, Y \times (-\infty, R_1]) \geq 3E_0/4, E(u \# r, Y \times [R_2, \infty)) \geq 3E_0/4.$$

By the gluing construction and the consequent elliptic estimates, $\mathcal{G}l_k(u, (tr)^{-1})$ also has an energy valley of length $2(tr)^{-1} - R_{u_1}^+ - R_{u_2}^- + o(1)$.

Similarly, $\mathcal{G}l_k(\tilde{u}(t), \tilde{r}(t)^{-1})$ has an energy valley of length $2\tilde{r}(t)^{-1} - R_{\tilde{u}_1(t)}^+ - R_{\tilde{u}_2(t)}^- + o(1)$.

We deduce

$$(7.15) \quad 2\tilde{r}(t)^{-1} - R_{\tilde{u}_1(t)}^+ - R_{\tilde{u}_2(t)}^- + o(1) = 2(tr)^{-1} - R_{u_1}^+ - R_{u_2}^- + o(1).$$

Since $\tilde{u}(t) \rightarrow u$ as $t \rightarrow 0$, we have

$$(7.16) \quad R_{\tilde{u}_1(t)}^+ \rightarrow R_{u_1}^+, R_{\tilde{u}_2(t)}^- \rightarrow R_{u_2}^-.$$

Combining (7.15) with (7.16) we infer that

$$\tilde{r}(t)^{-1} = (tr)^{-1} + o(1).$$

By Lemma 7.8 we then deduce that $u(t) = \tilde{u}(t)$, $\tilde{r}(t) = tr$ for small t . Now we can repeat the argument with a small, positive t_0 replacing $t = 0$. By extension and continuity, we conclude that $\tilde{u}(1) = u(1)$, $\tilde{r}(1) = r$, i.e. $\tilde{u} = u$, $\tilde{r} = r$.

Finally, Proposition 6.11 implies that $[\tilde{\mathcal{G}}l_k^T]$ is proper. Consequently, it is a homeomorphism. \square

Proof of Proposition 7.1. We define \mathbf{F}_k to be $[\tilde{\mathcal{G}}l_k^T]$. The previous lemmas immediately lead to the desired properties of \mathbf{F}_k . Note in particular that the smooth transition property (property (4)) is an easy consequence of Lemma 7.7 and Lemma 7.8. \square

Remark 7.12. All the above arguments extend straightforwardly to moduli spaces of transition trajectories as can easily be seen. Note that constant trajectories appear in the invariance proof. They have zero energy, and hence the energy valley argument in the proof of Lemma 7.10 does not apply to them. However, their moduli spaces are smoothly compact, namely the piecewise exponential convergence here is actually exponential convergence to smooth trajectories. Hence there is no need of using the energy valley argument for them.

Finally, we sketch the gluing process in terms of the fixed-end model. First, we need a presentation model for $\underline{\mathbf{M}}^0(p, q; \mathcal{SW}_0)$, where the underline means quotient under the twisted time translation action introduced in [24, Definition 6.16]. The twisted time translation action can be handled in a similar way to the time translation action for the temporal model. On the other hand, we can use the global Columb gauge to handle the based gauge action. Namely we consider the slice given by $d_\delta^*(A - A_0) = 0$. Here there is a delicate point when p or q is reducible. In this case, additional gauges in the isotropy group G_p or G_q (see [24, Section 4]) are involved. As a consequence, we can only obtain local slices instead of global slices. The remaining parts of the process are similar to the case of the temporal model. We omit the details. In comparison, we see two advantages of the temporal model approach. The first is the fact that it is canonical, while the fixed-end model involves the choice of \mathcal{SW}_0 . The second is that we have global slices and presentation models for the temporal model, while we only have local slices and presentation models for the fixed-end model.

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